Facoltà di Scienze Matematiche, Fisiche e Naturali
Ph.D Thesis in Mathematics, Sapienza Università di Roma
XXV ciclo

From microscopic dynamics to kinetic equations

Chiara Saffirio

Supervisor: Prof. Mario Pulvirenti

Ai miei genitori.

Acknowledgments

First of all I would like to express my gratitude to Prof. Mario Pulvirenti, for his support in my Ph.D study and research, for his knowledge, motivation and enthusiasm, for the great influence he has had not only in my professional growth but also on my character.

I am also grateful to Prof. Giovanni Borgioli, Prof. Carlo Marchioro and Prof. Rossana Marra for agreeing to serve as a committee members in my defense examination.

I wish to underline that this dissertation would not have been possible without the professional and human support of Prof. Laurent Desvillettes and Dr. Evelyne Miot, with whom I collaborated for a large part of my Ph.D. I have been privileged to known and collaborate with many other people: Prof. Alexander V. Bobylev, Prof. Carlo Boldrighini, Dr. Antonella Marchesiello, E. Rossi and Dr. Sergio Simonella. It has been a real honor for me to work with all of them.

I am obliged to the Mathematical Physics group in Roma La Sapienza, and in particular to Dr. Giada Basile, Dr. Guido Cavallaro and Dr. Gianluca Panati for giving me many pratical advices. I also would like to thank the GREFI-MEFI for having supported my Parisian time and all the French mathematicians with whom I had the honor to interact in that period.

I am obliged to many of my colleagues and friends, who supported me in these three years. In particular, I am deeply grate to Giuseppe, Renato, Graziella and Federica for useful discussions and true friendship. I would like also to thank all the Ph.D students in the mathematical department in Roma La Sapienza, Elena and Serena in Roma 3, my friends and Andrea for his helpful hints.

Finally I wish to express my gratitude to my parents, to whom this thesis is dedicated.

Contents

Chapter 1

General presentation: scaling limits in Kinetic Theory

1.1 Introduction

The purpose of Kinetic Theory is to describe systems made of a large number of components, which we will suppose in the sequel to be identical particles. The difficulty in the mathematical study of these systems relies in the huge number of particles; however the key point in kinetic theory is that we are not interested in the detailed analysis of the motion of each particle, but in the collective behavior of the system. Kinetic Theory studies methods to simplify the model in order to obtain a reduced picture preserving all the interesting physical informations of the many particle system. These methodologies make use of the limiting procedure from microscopic description of a many particle system based on the fundamental laws of mechanics (classical or quantum) to a kinetic picture.
To handle this problem, the idea is to use the statistical description of the many particle system (for example a gas or a plasma) given by a distribution function f in the particle phase space; more precisely the kinetic model associated to a given system in $\mathbb{R}^3$ is obtained by means of the evolution equation of a nonnegative function $f(t, x, v)$ defined on $\mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3$. The variables t, x and v represent respectively time, position and velocity. The time evolution of f is a priori described by the Liouville equation and its analysis should retain all the features observed at a macroscopic level. This is possible thanks to the claim that starting from a system at time $t = 0$, it is possible to recover its evolution using the law of classical and quantum mechanics. In this thesis we will focus only on the classical dynamics. The idea relies on the assumption that qualitative changes in the laws of mechanics are not necessary to understand the reason why the collective behavior of the system seems to contradict them (here we refer to the famous irreversibility paradox[1]). This controversial problem was largely

[1]Here we refer to the Loschmidt's paradox, which we briefly report here for sake of completeness. Let us consider the evolution of a gas in the time interval $[0, T]$ and imagine that at time T we are able to reverse all

studied in the last two centuries. We will emphasize this concept in the following, in particular underlining that the choice of initial data is a crucial point since in it the main probabilistic tool is hidden and it justifies somehow the apparent deviations (and contradiction) from classical dynamics.
We observe that at a fixed time t, the measure $f(t,x,v)\,dx\,dv$ represents the probability distribution of particles. At this point the common sense suggests that in a bounded subset of the physical space the integral of f is finite (see for instance [L-75]), so that the minimal assumption on the density function is that $f(t,\cdot,\cdot) \in L^1_{loc}(\mathbb{R}^3; L^1(\mathbb{R}^3))$ for all $t \in \mathbb{R}^+$. Here we are assuming that the system is made of so many particles that it can be represented as a continuum and this is the reason why the distribution function stands for an approximation of the true density on a macroscopic scale as well as it constitutes a lack of information in the knowledge of the true positions of particles.
If the collisions between particles in a gas were negligible, each particle would represent a closed subsystem and the time evolution of the distribution function would be

$$\frac{df}{dt} = 0 \; , \tag{1.1.1}$$

where d/dt stands for the material derivative, i.e.

$$\partial_t f + v \cdot \nabla_x f = 0 \; ,$$

if there are no external forces. This is the case of free motion.
If a force appears (namely $F = -\nabla_x \Phi$, where Φ is the internal interaction potential), then the evolution equation (1.1.1) becomes

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0 \; . \tag{1.1.2}$$

This is the case of the Vlasov equation, representing a collision-less plasma, where the force F is self induced, depending on the interaction potential and on the solution itself:

$$F(t,x) = (-\nabla_x \Phi * \rho)(t,x)$$

being $\rho(t,x) = \int dv\, f(t,x,v)$ the spatial density.
If we take into account the collisions, the Liouville equation (1.1.1) changes. It is natural to introduce an operator $Q(f,f)$, called the *collision integral*, describing the speed of variation of the distribution function after collisions, so that

$$\partial_t f + v \cdot \nabla_x f = Q(f,f) \; . \tag{1.1.3}$$

Equation (1.1.3) is a prototype of what is generally called *kinetic equation*.

---

velocities of the particles composing the gas. From a microscopic point of view we should be able to recover the same evolution backwards in time and to reach the initial configuration, but from a macroscopic point of view it is not so because of the entropy dissipation (see [CIP] or [V-02]).

The basic equation of kinetic theory is the celebrated *Boltzmann equation* (or *kinetic equation for dilute gases*):

$$\partial_t f + v \cdot \nabla_x f = \int dw \int_{S^2_-} d\nu B(v-w;\omega)[f(x,v')f(x,w') - f(x,v)f(x,w)] \tag{1.1.4}$$

with $B(v-w;\omega)$ a suitable function of the relative velocities $(v-w)$ and ω, the unit vector bisecting the angle between the incoming and the outgoing relative velocities. It is a non-linear integro-differential equation, describing the time evolution of the density of a dilute (monoatomic) gas. It was the first kinetic equation in the history of statistical mechanics and it was established by L. Boltzmann in 1872 ([B]). Equation (1.1.4) has been largely investigated because of its interest both for fundamental theory and practical applications. The remarkable fact is that, in the attempt to conciliate the Newton laws with the second principle of thermodynamics, Boltzmann was able to construct an equation expressing mass, momentum and energy conservations, but also the trend to thermal equilibrium. In particular, if f is a solution to the Boltzmann equation (1.1.4), the following conservations laws hold:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dv\ f(t,x,v) = 0\ , && \text{conservation of the total mass;}\\
&\frac{d}{dt}\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dv\ v_i\, f(t,x,v) = 0\ , \quad i=1,2,3 && \text{conservation of the total momentum;}\\
&\frac{d}{dt}\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dv\ \frac{|v|^2}{2} f(t,x,v) = 0 && \text{conservation of the total energy.}
\end{aligned} \tag{1.1.5}$$

Equations (1.1.5) are easily checked by using the explicit form of the collision operator and standard manipulations.
Moreover, we introduce the *H–functional*, which represents the *entropy* of the system (with the opposite sign with respect to the physical entropy):

$$H(f(t,\cdot,\cdot)) = \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dv\ f(t,x,v)\log\left(f(t,x,v)\right)\ . \tag{1.1.6}$$

Boltzmann observed that, if f is a solution to (1.1.4), the time derivative of the H–functional is non increasing, indeed

$$\begin{aligned}
&\frac{dH}{dt}(f(t,\cdot,\cdot)) = \int dx \int dv\, Q(f,f)\log f = \\
&= -\frac{1}{4}\int dx \int dv \int dw \int_{S^2_-} d\nu B(v-w,\omega)\times \\
&\qquad \times \left(f(t,x',v')f(t,x',w') - f(t,x,v)f(t,x,w)\right)\log\frac{f(t,x',v')f(t,x',w')}{f(t,x,v)f(t,x,w)} \le 0\ ,
\end{aligned} \tag{1.1.7}$$

where in the last line we used that the function $(x,v) \longmapsto \left((x-y)\log\frac{x}{y}\right)$ is non negative.

Inequality (1.1.7) is the celebrated Boltzmann's *H–Theorem*, which states that the entropy in non increasing in time.

In Chapter 2 we will report the paper [PSS], in which we discuss the problem related to the derivation of the Boltzmann equation from a N-particle system, in the spirit of the well known paper by Lanford ([L-75], for a system of hard–spheres, and [K-75]). Here we propose a rigorous derivation in the case in which the interaction is given by a smooth short range positive potential. More precisely, in [PSS] we show that, considering a classical system of point particles interacting by means of a short range potential and performing the *low–density limit* (or Boltzmann–Grad limit: see Section 1.3), the system behaves, for short times, as predicted by the associated Boltzmann equation.

When a long range interaction appears, it is not clear whether the Boltzmann equation is a suitable model. In particular, in the case of Coulomb interactions, the collision integral becomes divergent at large distances among particles and equation (1.1.4) makes no sense. For this reason, L. D. Landau in 1936 ([L-36]) proposed the following kinetic equation, called the *Landau equation*:

$$\partial_t f + v \cdot \nabla_x f = \int dw \nabla_v \left[a(v-w) \left(\nabla_v - \nabla_w \right) f(v) f(w) \right] \ , \tag{1.1.8}$$

being $a(v-w)$ a matrix of the form

$$a(v-w) = \frac{A}{|v-w|} \frac{\left(|v-w|^2 Id - (v-w) \otimes (v-w) \right)}{|v-w|^2}, \tag{1.1.9}$$

where $A > 0$ is a suitable constant.

The Landau equation (1.1.8) retains conservation laws (1.1.5) and, choosing the H-functional as in (1.1.6), an equivalent H-Theorem holds:

$$\begin{aligned} &\frac{dH}{dt}(f(t,\cdot,\cdot)) = \\ &\quad = -\frac{1}{2} \int dx \int dv \int dw \ a(v-w) \ f(t,x,v) \ f(t,x,w) \left| \frac{\nabla_v f(t,x,v)}{f(t,x,v)} - \frac{\nabla_w f(t,x,w)}{f(t,x,w)} \right|^2 \leq 0 \ . \end{aligned} \tag{1.1.10}$$

This equation is largely used in plasma physics and the mathematical theory is at the very beginning. Indeed very little is known about the well-posedness problem and the derivation from particle system. In Chapter 3 we will propose an attempt to derive the Landau equation from a system of particles interacting by means of a smooth short range potential, reporting the paper [BPS], in which we perform the *weak–coupling limit* (see Sections 1.3 and 1.4) to pass from microscopic to macroscopic dynamics. The result is very preliminary, since we are able to give only a rigorous consistency proof.

Since the mathematical problem linked to the derivation of the Landau equation from a deterministic particle system seems to be very difficult to handle, in Chapter 4 we will present a result concerning the derivation of the Landau equation from a stochastic model, which plays the same role of the Kac model [K] for the Boltzmann equation. After a brief introduction, in Sections 4.2 and 4.3, we review the well known result obtained by Kac in 1956 and explain how it is possible to obtain the Landau collision operator from the Boltzmann integral, performing the *grazing collision limit*, exactly as Landau did in 1936. In Section 4.4 we report the paper [MPS] where a Kac model for the Landau equation is obtained.

Finally, Chapter 5 is devoted to the study of the Vlasov–Poisson equation, which is the usual name for equation (1.1.2) when the interaction potential is Coulomb. By analogy with the previous Chapters, in Section 5.2 we recall that the sailing limit in which the Vlasov equation is expected to hold is the mean–field limit. In particular, we are not interested in the derivation problem, but we focus on the Cauchy problem for the three–dimensional repulsive Vlasov–Poisson system in presence of a point charge (also called the *plasma–charge model*). In Section 5.3 we report the work in progress [DMS], in which - using the well known results [LP] and [MMP] - we give an existence result for a quite large class of initial data.

The remaining of the present Chapter is devoted to introduce the mathematical objects and some important notions (for instance *propagation of chaos*) we will use in the following.

1.2 Newton equations

We consider a system of N particles in the whole space $\mathbb{R}^3$, interacting by means of a two–body potential Φ. A state of the N–particle system is denoted by $\mathbf{z}_N = (\mathbf{q}_N, \mathbf{v}_N) = (q_1, \dots, q_N, v_1, \dots, v_N) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$, where q_i and v_i are respectively the position and the velocity of particle i, for $i = 1, \dots, N$.

It is reasonable from a physical point of view to assume that particles in the phase space $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$ are identical; this means that we consider the configuration $\mathbf{z}_N$ belonging to the quotient space

$$\mathcal{S} := (\mathbb{R}^3 \times \mathbb{R}^3)^N / S_N \ ,$$

where S_N is the permutation group.

Assuming that the mass of the identical particles is equal to one for sake of simplicity, the N–particle Hamiltonian is

$$H_N(\mathbf{q}_N, \mathbf{v}_N) = \sum_{i=1}^{N} \frac{v_i^2}{2} + U(\mathbf{q}_N) \ , \tag{1.2.1}$$

where the first term in the r.h.s. of eq.n (1.2.1) is the total kinetic energy of the N-particle system and the second term describes the interaction among particles by means of a potential energy which is the sum of all the two–body interactions:

$$U(\mathbf{q}_N) = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \Phi(q_i - q_j) \ .$$

Thanks to the identical nature of particles, the Hamiltonian (1.2.1) is symmetric with respect to any permutation of particles, i.e.

$$H(\mathbf{q}_N, \mathbf{v}_N) = H(\sigma(\mathbf{q}_N), \sigma(\mathbf{v}_N)) \ , \qquad \forall (\mathbf{q}_N, \mathbf{v}_N) \in (\mathbb{R}^{3N} \times \mathbb{R}^{3N}) \tag{1.2.2}$$

where $\sigma \in S_N$ is a given permutation of N elements. It follows that it is enough to choose $(\mathbf{q}_N, \mathbf{v}_N)$ in the quotient space $\mathcal{S}$.
Moreover, we observe that the Hamiltonian does not depend explicitly on the time variable, so that the hamiltonian system is conservative.
Fixed an initial configuration $(\mathbf{q}_N, \mathbf{v}_N) \in \mathcal{S}$, the time evolution of the N–particle system associated to eq.n (1.2.1) is given by the Newton equations, i.e. the dynamics is governed by the following system of ordinary differential equations

$$\begin{cases} \dot{q}_i(\tau) = v_i(\tau) \ , \\ \\ \dot{v}_i(\tau) = \sum_{\substack{j=1 \\ j \neq i}}^{N} F(q_i(\tau) - q_j(\tau)) \ , \quad i = 1, \dots, N \end{cases} \tag{1.2.3}$$

where $F(q_i(\tau) - q_j(\tau))$ is the force acting on particle i due to particle j at time τ; more precisely $F(q_i - q_j) = -\nabla\Phi(q_i - q_j)$. If we assume that the potential is twice differentiable and bounded with bounded derivatives, i.e. $\Phi \in \mathcal{C}^2_b(\mathbb{R}^3)$, there exists a unique flow S^τ, solution to (1.2.3).
Since we are interested in a statistical description of the system when the number of particles becomes huge, we consider on the phase space $(\mathbb{R}^{3N} \times \mathbb{R}^{3N})$ the N–particle probability distribution $f_0^N(\mathbf{q}_N, \mathbf{v}_N) d\mathbf{q}_N \mathbf{v}_N$ at time zero. In particular the probability density f_0^N has the following properties:

(i) $f_0^N(\mathbf{q}_N, \mathbf{v}_N) \geq 0$, for all $(\mathbf{q}_N, \mathbf{v}_N) \in (\mathbb{R}^3 \times \mathbb{R}^3)$;

(ii) $\int_{\mathbb{R}^{3N}} d\mathbf{q}_N \int_{\mathbb{R}^{3N}} d\mathbf{v}_N f_0^N(\mathbf{q}_N, \mathbf{v}_N) = 1$;

(iii) $f_0^N(\mathbf{q}_N, \mathbf{v}_N)$ is symmetric in the exchange of particles.

Properties (i) and (ii) are just the definition of probability density, while (iii) is a consequence of (1.2.2).
Thanks to the Liouville Theorem, the Hamiltonian flow $S^\tau(\mathbf{q}_N, \mathbf{v}_N)$ associated to (1.2.3) is such that

$$f^N(\tau, \mathbf{q}_N, \mathbf{v}_N) = f_0^N(S^{-\tau}(\mathbf{q}_N, \mathbf{v}_N)) \ . \tag{1.2.4}$$

This ensures that, if f_0^N is a N–particle probability density, its evolution at time $\tau > 0$ is a probability density too, which preserves the symmetry property (iii).
Let us denote by $f^N(\tau) = f^N(\tau, \mathbf{q}_N, \mathbf{v}_N)$ the time evolution of the probability density; it is obtained by solving the Cauchy problem associated to the Liouville equation

$$\partial_\tau f^N(\tau) + \mathbf{v}_N \cdot \nabla_{\mathbf{q}_N} f^N(\tau) = \nabla_{\mathbf{q}_N} U \cdot \nabla_{\mathbf{v}_N} f^N(\tau) \tag{1.2.5}$$

with initial datum f_0^N. In eq.n (1.2.5) we used the short notations $\mathbf{v}_N\cdot\nabla_{\mathbf{q}_N}$ and $\nabla_{\mathbf{q}_N}U\cdot\nabla_{\mathbf{v}_N}$ to indicate respectively $\sum_{i=1}^N v_i\cdot\nabla_{q_i}$ and $\sum_{i=1}^N \nabla_{q_i}U\cdot\nabla_{v_i}$.
We observe that eq.n (1.2.5) follows easily by the Liouville Theorem (1.2.4) and eq.ns (1.2.3). Indeed let $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}^{3N}\times\mathbb{R}^{3N})$ be a test function, smooth and complactly supported. On the one side

$$\begin{aligned}
&\frac{d}{d\tau}\int d\mathbf{q}_N\,d\mathbf{v}_N\,f_0^N(S^{-\tau}(\mathbf{q}_N,\mathbf{v}_N))\varphi(\mathbf{x}_N,\mathbf{v}_N)=\\
&\quad=-\int d\mathbf{q}_N\,d\mathbf{v}_N\,\left[\dot{\mathbf{q}}_N\cdot\nabla_{\mathbf{q}_N}f^N(\tau)+\dot{\mathbf{v}}_N\cdot\nabla_{\mathbf{v}_N}f^N(\tau)\right]\varphi(\mathbf{x}_N,\mathbf{v}_N)=\\
&\quad=-\int d\mathbf{q}_N\,d\mathbf{v}_N\,\left[\mathbf{v}_N\cdot\nabla_{\mathbf{q}_N}f^N(\tau)+\nabla_{\mathbf{q}_N}U\cdot\nabla_{\mathbf{v}_N}f^N(\tau)\right]\varphi(\mathbf{x}_N,\mathbf{v}_N)\;;
\end{aligned}$$

on the other side, by (1.2.4),

$$\frac{d}{d\tau}\int d\mathbf{q}_N\,d\mathbf{v}_N\,f^N(t,\mathbf{q}_N,\mathbf{v}_N)\varphi(\mathbf{x}_N,\mathbf{v}_N)=\int d\mathbf{q}_N\,d\mathbf{v}_N\,\partial_\tau f^N(\tau,\mathbf{q}_N,\mathbf{v}_N)\varphi(\mathbf{x}_N,\mathbf{v}_N)\;.$$

The natural starting problem is to understand whenever the flow $S^\tau(\mathbf{q}_N,\mathbf{v}_N)$ associated to (1.2.3) exists and if it is unique. It is well known by classical theory of ordinary differential equations that the existence and uniqueness of the flow associated to (1.2.3) is strictly linked to the regularity of the interaction potential; for instance we know that if $\Phi\in\mathcal{C}_b^2(\mathbb{R}^3)$, for all initial states of the system $(\mathbf{q}_N,\mathbf{v}_N)\in(\mathbb{R}^{3N}\times\mathbb{R}^{3N})$ there exists a unique flow $S^\tau(\mathbf{q}_N,\mathbf{v}_N)$ associated to (1.2.3), i.e. the dynamics is well defined everywhere.

1.3 Low density and weak–coupling limits

We are interested in a situation in which the number of particles N is very large, so it is natural to investigate the limit $N\to\infty$. A natural way to do this is to pass from a microscopic description to a macroscopic one, by means of a scaling limit.
In the present Section we will describe two types of scaling limits: the low–density and the weak–coupling limits.
We consider the N–particle system introduced in Section 1.2, obeying to the usual Newton eq.ns (1.2.3), and a small scale parameter $\varepsilon>0$ which expresses the ratio between the macro and the micro unites.
If we are interested in the description of a rarefied gas, it is convenient to rescale eq.ns (1.2.3) in terms of macroscopic variables

$$t=\varepsilon\tau,\qquad x_i=\varepsilon q_i,\quad\forall\; i=1,\dots,N$$

whenever the physical variables of interest are varying on such scales and are almost constant on the microscopic scale.

The dynamics is described by the rescaled equations

$$\begin{cases} \dot{x}_i(t) = v_i(t) \ , \\ \\ \dot{v}_i(t) = \frac{1}{\varepsilon} \sum_{\substack{j=1 \\ j \neq i}}^{N} F\left(\frac{x_i(t)-x_j(t)}{\varepsilon}\right) \ , \quad i = 1, \dots, N \end{cases} \tag{1.3.1}$$

We notice that in order to have a kinetic picture for a rarefied gas, a tagged particle (say particle i) must undergo a finite number of collisions in a macroscopic unit time. As a consequence, the density $N\varepsilon^3$ must vanish, i.e. $N\varepsilon^3 \to 0$. More precisely, it should behave as $\mathcal{O}(\varepsilon^2)$; indeed, assuming the interaction length of the potential Φ to be one (i.e. $\Phi(r) = 0$ if $r \geq 1$), we consider the tube spanned by particle i and such a number is finite. In other words, the limit situation in which the gas is rarefied, but the number of collisions that each particle undergoes per unit time is not negligible is well described by

$$N \longrightarrow \infty \ , \qquad \varepsilon \longrightarrow 0 \ , \tag{1.3.2}$$

with the constraint

$$N\varepsilon^2 = \lambda^{-1} \tag{1.3.3}$$

where $\lambda > 0$ is the mean–free path. The scaling (1.3.2)–(1.3.3) takes into account the low–density of the gas and for this reason it is called *low–density limit*.

As a consequence of the scaling, the probability that two tagged particles (say particle i and particle j) collide is negligible since it is of order $\mathcal{O}(\varepsilon^2)$. In fact, if we assume that particles are balls with diameters ε, the probability of the event $\{$*the couple* (i, j) *collides*$\}$ is of order of the surface of the ball, i.e. $\mathcal{O}(\varepsilon^2)$. However the probability that a given particle performs a collision with any one of the remaining $N - 1$ particles is not negligible, indeed it is $N\varepsilon^2$, that is $\mathcal{O}(1)$ thanks to (1.3.3).

As we shall see in the next Section by heuristic arguments and in Chapter 2 in detail, the low–density limit is the scaling in which the Boltzmann equation (1.1.4) is expected to hold. We remark that in the case of hard–spheres, namely for a N–particle system of elastic and non overlapping balls, the low–density limit is completely equivalent to the well known Boltzmann–Grad limit, so called by the names of physicists that stated heuristically and rigorously the scaling (1.3.3).

Although the Boltzmann equation is expected to be a good description for the time evolution of the probability density of a rarefied gas, a natural question is whether it is possible to obtain a kinetic picture for a dense gas.

If we consider a situation in which the number of particles is very large and the interaction quite moderate, we can perform the so–called *weak–coupling limit*. We deal with the usual N–particle system introduced at the beginning of the present Section, whose dynamics is described by (1.2.3). As already done in the case of a low–density regime, we introduce a small scale parameter $\varepsilon > 0$ and we rescale (1.2.3) in terms of the macroscopic variables

$$t = \varepsilon\tau, \qquad x_i = \varepsilon q_i, \quad \forall \ i = 1, \dots, N \ .$$

Moreover we rescale the potential

$$\Phi(\cdot) \longrightarrow \sqrt{\varepsilon}\Phi(\cdot) \ ,$$

expressing the weakness of the interaction. The rescaled system is

$$\begin{cases} \dot{x}_i(t) = v_i(t) \ , \\ \\ \dot{v}_i(t) = -\frac{1}{\sqrt{\varepsilon}} \sum_{\substack{j=1 \\ j\neq i}}^{N} \nabla\Phi\left(\frac{x_i(t)-x_j(t)}{\varepsilon}\right) \ , \quad i = 1, \dots, N \end{cases} \tag{1.3.4}$$

where we used that the inter particle force is conservative since $F = -\nabla\Phi$.
To take into account the high density of the gas, we assume that $N\varepsilon^3$ is of order one, i.e. $N\varepsilon^3 = \lambda^{-1} > 0$. In this contest, the probability that two given particles interact vanishes in the limit $\varepsilon \to 0$ because of the weakness of the interaction. Indeed two particles can interact, but the collision has a small effect and the probability vanishes in the limit.
As we shall see in the next Section by heuristic arguments and rigorously in Chapter 3, the kinetic equation that is expected in the weak–coupling limit is the Landau equation (1.1.8).

1.4 From particle systems to kinetic equations: heuristic derivation

Both the Newton and the Liouville equations are difficult to deal with in the regime of N large. For this reason, in 1946 the physicists Bogolyubov, Born, Green, Kirkwood and Yvon introduced a reduced description of the N–particle system based on the asymptotic study -in a sense that we do not precise here[2]- of the j–particle marginal probability density. More precisely, they introduced the j–particle marginals

$$\begin{cases} f_j^N(\tau, \mathbf{q}_j, \mathbf{v}_j) = \int dq_{j+1}dq_{j+2}\dots dq_N \int dv_{j+1}dv_{j+2}\dots dv_N f^N(t, \mathbf{q}_N, \mathbf{v}_N) \ , & j = 1, \dots, N \\ \\ f_j^N(\tau, \mathbf{q}_j, \mathbf{v}_j) = 0 \ , & j > N \end{cases} \tag{1.4.1}$$

where f^N is the N–particle joint probability density. In particular, (1.4.1) expresses the probability density of a group of j particles arbitrarily chosen among the N particles at time τ. We observe that $f_N^N(\tau, \mathbf{q}_N, \mathbf{v}_N) = f^N(\tau, \mathbf{q}_N, \mathbf{v}_N)$.
In order to obtain an equation for the j–particle marginal $f_j^N(\tau, \mathbf{q}_j, \mathbf{v}_j)$, we integrate the Liouville equation (3.3.4) with respect to the $N - j$ remaining variables $\mathbf{z}_j^N = q_{j+1}, \ q_{j+2}, \ \dots, \ q_N,$ $v_{j+1}, \ v_{j+2}, \ \dots, \ v_N$ (in the following, we will use the notation $\mathbf{q}_j^N = q_{j+1}q_{j+2}\dots q_N$ and

[2]When the number of particles becomes huge, the asymptotic we are looking at depends on the phenomena we want to describe. For a rarefied gas, it will be the low–density limit; for a dense gas, the weak–coupling limit, as mentioned in Section 1.3.

$\mathbf{v}_j^N = v_{j+1}v_{j+2}\dots v_N$ for sake of brevity) and we obtain

$$\frac{\partial}{\partial\tau} f_j^N = \mathcal{L}_j f_j^N + (N-j) C_{j+1} f_{j+1}^N \ , \qquad j = 1, \dots, N \tag{1.4.2}$$

where $\mathcal{L}_j$ is the Liouville operator reduced to a j–particle subsystem and

$$C_{j+1} f_{j+1}^N(\tau, \mathbf{q}_j, \mathbf{v}_j) = \sum_{i=1}^{j} \int dq_{j+1} dv_{j+1} \nabla_{q_i} \Phi(q_i - q_{j+1}) \cdot \nabla_{v_i} f_{j+1}^N(\tau, \mathbf{q}_j, \mathbf{v}_j) \ , \tag{1.4.3}$$

for $j < N$ and $C_{N+1} = 0$. Indeed

$$\int d\mathbf{q}_j^N d\mathbf{v}_j^N \left(\frac{\partial}{\partial\tau} + \mathbf{v}_N \cdot \nabla_{\mathbf{q}_N}\right) f^N(\tau, \mathbf{q}_N, \mathbf{v}_N) = \left(\frac{\partial}{\partial\tau} + \sum_{i=1}^{j} v_i \cdot \nabla_{q_i}\right) f_j^N(\tau, \mathbf{q}_j, \mathbf{v}_j) \ , \tag{1.4.4}$$

$$\begin{aligned}
\int d\mathbf{q}_j^N d\mathbf{v}_j^N \nabla_{\mathbf{q}_N} U \cdot \nabla_{\mathbf{v}_N} f^N &= \sum_{i=1}^{N} \int d\mathbf{q}_j^N d\mathbf{v}_j^N \nabla_{q_i} U \cdot \nabla_{v_i} f^N = \\
&= \sum_{i=1}^{N} \sum_{\substack{k=1 \\ k\neq i}}^{N} \int d\mathbf{q}_j^N d\mathbf{v}_j^N \nabla_{q_i} \Phi(q_i - q_k) \cdot \nabla_{v_i} f^N = \\
&= \sum_{i=1}^{j} \sum_{\substack{k=1 \\ k\neq i}}^{N} \int d\mathbf{q}_j^N d\mathbf{v}_j^N \nabla_{q_i} \Phi(q_i - q_k) \cdot \nabla_{v_i} f^N + \\
&+ \sum_{i=j+1}^{N} \sum_{\substack{k=1 \\ k\neq i}}^{N} \int d\mathbf{q}_j^N d\mathbf{v}_j^N \nabla_{q_i} \Phi(q_i - q_k) \cdot \nabla_{v_i} f^N \ .
\end{aligned} \tag{1.4.5}$$

The last term in the above equation vanishes because of the integration; the first term could be written as follows:

$$\begin{aligned}
&\sum_{i=1}^{j} \sum_{\substack{k=1 \\ k\neq i}}^{N} \int d\mathbf{q}_j^N d\mathbf{v}_j^N \nabla_{q_i} \Phi(q_i - q_k) \cdot \nabla_{v_i} f^N = \\
&= \sum_{i=1}^{j} \sum_{\substack{k=1 \\ k\neq i}}^{j} \int d\mathbf{q}_j^N d\mathbf{v}_j^N \nabla_{q_i} \Phi(q_i - q_k) \cdot \nabla_{v_i} f^N + \sum_{i=1}^{j} \sum_{k=j+1}^{N} \int d\mathbf{q}_j^N d\mathbf{v}_j^N \nabla_{q_i} \Phi(q_i - q_k) \cdot \nabla_{v_i} f^N = \\
&= \sum_{i=1}^{j} \sum_{\substack{k=1 \\ k\neq i}}^{N} \int d\mathbf{q}_j^N d\mathbf{v}_j^N \nabla_{q_i} \Phi(q_i - q_k) \cdot \nabla_{v_i} f_j^N + \\
&+ \sum_{i=1}^{j} (N-j) \int dq_{j+1} dv_{j+1} \nabla_{q_i} \Phi(q_i - q_{j+1}) \cdot \nabla_{v_i} f_{j+1}^N \ ,
\end{aligned} \tag{1.4.6}$$

where we used the symmetry (in the exchange of particles) of the probability density to obtain a sum of $(N-j)$ equal contributions.

The meaning of the hierarchy (1.4.2) is the following: the time evolution of the j–particle probability density f_j^N is linked to the j–particle Liouville operator, which represents the interaction of the first j particles among themselves, and to the C_{j+1} operator, which depends on the interaction of the first j particles with the remaining $N-j$ particles.
We observe that when $j = N$ we recover exactly the Liouville equation.
The hierarchy (1.4.2) is called BBGKY (Bogolyubov, Born, Green, Kirkwood and Yvon) because of the names of the physicists who introduced it.
In the present Section we will use the BBGKY hierarchy to pass heuristically from the Hamiltonian N–particle system described in Section 1.2 to an appropriate kinetic equation by means of opportune scaling limits defined in Section 1.3.
To simplify the model, we consider a system of N identical particles of radius ε in the whole space and we suppose that the interactions among particles are elastic collisions, so that they cannot overlap and they change instantaneously their velocity according to the energy and momentum conservations. More precisely, if two particles (say particle i and particle j) collide with velocities v_i and v_j respectively, the pre–collisional velocities are

$$\begin{cases} v_i' = v_i - \nu[\nu \cdot (v_i - v_j)] \; , \\ v_j' = v_j + \nu[\nu \cdot (v_i - v_j)] \; ; \end{cases} \tag{1.4.7}$$

and ν is the unit vector indicating the direction of the line linking the two particles. The dynamics is driven in the following way: a tagged particle moves freely up to the first time in which it performs an elastic collision and changes velocity instantaneously according to (1.4.7). The procedure goes on iteratively. We notice that triple collisions are negligible because they are unlikely.
Boltzmann heuristic argument is the following: consider a test particle and denote by f the probability density associated to it; the evolution equation of the tagged particle we have considered is

$$\frac{\partial}{\partial t} f + v \cdot \nabla f = Coll \tag{1.4.8}$$

where $Coll$ denotes the effect that collisions produce on the variation of the probability density f. We observe that the operator $Coll$ should consist of two parts, a gain part denoted by G and a loss part L, due to the fact that they give respectively a positive or a negative contribution to the variation of f because of the collisions.
In particular, $Ldxdvdt$ is the probability that the particle disappears from the cell $dxdv$ of the phase space because of a collision in the time interval $(t, t+dt)$ and $Gdxdvdt$ is the probability that the particle appears in the same cell in the same time interval.
Boltzmann's argument is based on the following considerations: we focus on a tagged particle (x_1, v_1) in the phase space and we consider the sphere of centre x_1 and radius ε; a point on the surface is determined by $x_1 + \varepsilon\nu$, where $\nu \in S^2$, being S^2 the unit sphere in $\mathbb{R}^3$ centered in x_1. Let (x_2, v_2) be another particle in the phase space; we look at the cylinder with base area $\varepsilon^2 d\nu$ and height $|(v_2 - v_1)|dt$ along the direction $(v_2 - v_1)$ and we see that the contribution of

particle 2 to the loss term L depends on the sign of the scalar product between the relative velocity $(v_2 - v_1)$ and ν: if $(v_2 - v_1) \cdot \nu \leq 0$, then particle 2 can collide with particle 1 in the time interval dt so that it can contribute to L; if not, particle 2 do not contribute. These contributions are equivalent to the probability of finding a particle in the cylinder knowing the presence of particle 1 in x_1, i.e.

$$f_2(x_1, x_1 + \varepsilon\nu, v_1, v_2)|\,(v_2 - v_1) \cdot \nu|\,\varepsilon^2\, d\nu\, dv_2\, dt\ .$$

If we integrate in the v_2 and ν variables, the total contribution given to the loss term L by each single particle is

$$\varepsilon^2 \int dv_2 \int_{S^2_-} d\nu f_2(x_1, x_1 + \varepsilon\nu, v_1, v_2)|(v_2 - v_1) \cdot \nu|$$

where $S^2_- = \{\nu \in S^2 \mid (v_2 - v_1) \cdot \nu < 0\}$. As a consequence, taking into account that particles are identical, the total contribution to the loss term is

$$L = (N-1)\varepsilon^2 \int dv_2 \int_{S^2_-} d\nu f_2(x_1, x_1 + \varepsilon\nu, v_1, v_2)|(v_2 - v_1) \cdot \nu|\ . \tag{1.4.9}$$

In the same way, the contribution to the gain term is

$$G = (N-1)\varepsilon^2 \int dv_2 \int_{S^2_+} d\nu f_2(x_1, x_1 + \varepsilon\nu, v_1, v_2)|(v_2 - v_1) \cdot \nu|\ , \tag{1.4.10}$$

where $S^2_+ = \{\nu \in S^2 \mid (v_2 - v_1) \cdot \nu > 0\}$.
Therefore the collision operator is the sum of the two contributions:

$$Coll = (N-1)\varepsilon^2 \int dv_2 \int d\nu f_2(x_1, x_1 + \varepsilon\nu, v_1, v_2)|(v_2 - v_1) \cdot \nu|\ . \tag{1.4.11}$$

We notice that (1.4.8) is not a closed equation; indeed the knowledge of the two–particle probability density is necessary to solve the equation and to find the one–particle probability density; in the same manner the knowledge of f_2 depends on f_3 and so on. In order to find a solution to (1.4.8) it is indispensable to get a closed equation; to this end a key role is played by the Boltzmann's main assumption: the "Stosszahlansatz". Boltzmann's idea is that, if the gas is rarefied, two given particles are uncorrelated , i.e.

$$f_2(x_1, x_2, v_1, v_2) = f(x_1, v_1) f(x_2, v_2)\ . \tag{1.4.12}$$

At a first glance eq.n (1.4.12) seems to be false: it means that the positions and velocities of the particles are chosen randomly and independently, according to a profile f. This can be done in general only at time zero, since correlations are created as soon as a collision happens. Indeed, even if we assume (1.4.12) to be true at time zero, if the test particle (x_1, v_1) collides with particle (x_2, v_2), (1.4.12) cannot hold after the collision (since time creates correlations). However, thanks to the low density of the gas, assumption (1.4.12) is not completely unreasonable; in fact it should be true in some limit and the point is to find the right scaling. To this end, we observe that if $N\varepsilon^2 = \mathcal{O}(1)$ and f_2 is smooth, the gain term is order $\mathcal{O}(1)$. We notice also that the probability that two tagged particles collide is $\mathcal{O}(\varepsilon^2)$ and that

$$f_2(x_1, x_1 + \varepsilon\nu, v_1, v_2) = f_2(x_1, x_1 + \varepsilon\nu, v_1', v_2') \ , \tag{1.4.13}$$

where v_1' and v_2' are the pre–collisional velocities, according to (1.4.7). This suggests that assumption (1.4.12) makes sense and, performing the change of variables $\nu \to -\nu$ in the gain term, the collision operator appears as

$$Coll = (N-1)\varepsilon^2 \int dv_2 \int_{S^2_-} d\nu (v_1 - v_2) \cdot \nu [f(x_1, v_1')f(x_1 - \varepsilon\nu, v_2') - f(x_1, v_1)f(x_1 + \varepsilon\nu, v_2)] \ . \tag{1.4.14}$$

Heuristically, if f is smooth enough, in the low–density limit (see Section 1.3) the resulting equation is exactly the one obtained by Boltzmann:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \int dv_2 \int_{S^2_-} d\nu \ (v_1 - v_2) \cdot \nu [f(x_1, v_1')f(x_1, v_2') - f(x_1, v_1)f(x_1, v_2)] \ . \tag{1.4.15}$$

It is important to underline that eq.n (1.4.15) is not equivalent to the Hamiltonian dynamics from which it is derived. In fact it has a statistical nature and we stress that it is not time–reversal, thanks to H-Theorem. In particular, eq.n (1.4.12) (called *propagation of chaos* in a more general situation) implies an asymmetry in the time variable; indeed if pre–collisional velocities are uncorrelated, post–collisional velocities have to be correlated, creating an asymmetry between past and future. The key point is that the microscopic dynamics is reversible while the macroscopic one is irreversible, also if at a first glance it seems that we have used only deterministic tools[3].

The heuristic argument presented in this Section can be extended to a more general class of two–body potential, obtaining in the limit the classical formulation of the Boltzmann equation:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f) \ , \tag{1.4.16}$$

where

$$Q(f, f) = \int dv_2 \int_{S^2_-} d\nu B(v_1 - v_2; \omega)[f(x_1, v_1')f(x_1, v_2') - f(x_1, v_1)f(x_1, v_2)] \tag{1.4.17}$$

with $B(v_1 - v_2; \omega)$ a suitable function of the relative velocities and ω.
Of course, if we want to derive rigorously (1.4.16) from the hamiltonian dynamics, we have to take into account that the interaction time is not anymore a time instant if we do not consider the hard–sphere model and that the expression of the pre–collisional velocities in terms of

[3]Indeed the probabilistic meaning is hidden in the particular choice of the initial datum according to the so–called propagation of chaos at time zero.

the post–collisional ones is a priori a complicated function of the relative velocities and the impact parameters. We refer to Chapter 2 and its Appendix for a detailed explanation of the problem.

Starting from eq.n (1.4.2), it is possible to perform a different scaling limit to obtain, at least formally, the Landau equation that describes the time evolution of the probability density of a plasma, in which the density is high.
In particular, we are interested in a situation in which the number of particles N is very large and the interaction strength quite moderate. The system has a unitary density so that we assume $N = \varepsilon^{-3}$. In addition we look for a reduced or macroscopic description of the system. Namely if q and τ refer to the system seen in a microscopic scale, we rescale eq.n (1.2.3) in terms of the macroscopic variables

$$x = \varepsilon q \qquad t = \varepsilon\tau \tag{1.4.18}$$

whenever the physical variables of interest are varying on such scales and are almost constant on the microscopic scale. Remembering that we want to describe weakly interacting systems, we perform the weak–coupling limit (see Section 1.3) rescaling the potential according to:

$$\Phi \to \sqrt{\varepsilon}\Phi, \tag{1.4.19}$$

so that system (1.2.3), in terms of the (x, t) variables, becomes:

$$\begin{cases} \frac{d}{dt}x_i = v_i \\ \\ \frac{d}{dt}v_i = -\frac{1}{\sqrt{\varepsilon}}\sum_{\substack{j=1\dots N:\\ j\neq i}} \nabla\Phi(\frac{x_i-x_j}{\varepsilon}) = \frac{1}{\sqrt{\varepsilon}}\sum_{\substack{j=1\dots N:\\ j\neq i}} F(\frac{x_i-x_j}{\varepsilon}) \;. \end{cases} \tag{1.4.20}$$

A statistical description of the above system passes through the introduction of a probability distribution on the phase space of the system. Let $f^N = f^N(t, \mathbf{x}_N, \mathbf{v}_N)$ be, as usual, a symmetric (in the exchange of variables) probability density. Then from eq.n (1.4.20) we obtain the following Liouville equation

$$(\partial_t + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i}) f_N^N(t, \mathbf{x}_N, \mathbf{v}_N) = \frac{1}{\sqrt{\varepsilon}}\big(T_N^\varepsilon f_N^N\big)(t, \mathbf{x}_N, \mathbf{v}_N). \tag{1.4.21}$$

Here we have introduced the operator

$$\big(T_N^\varepsilon f_N^N\big)(t, \mathbf{x}_N, \mathbf{v}_N) = \sum_{0<k<\ell\leq N} \big(T_{k,\ell}^\varepsilon f_N^N\big)(t, \mathbf{x}_N, \mathbf{v}_N), \tag{1.4.22}$$

with

$$T_{k,\ell}^\varepsilon f_N^N = \nabla\Phi(\frac{x_k - x_\ell}{\varepsilon}) \cdot (\nabla_{v_k} - \nabla_{v_\ell}) f_N^N. \tag{1.4.23}$$

To investigate the limit $\varepsilon \to 0$ it is convenient to introduce the BBGKY hierarchy for the j-particle distributions $f_j^N(\mathbf{x}_j, \mathbf{v}_j)$, for $j = 1. \dots, N-1$, defined in (1.4.1). Such a hierarchy is obtained by means of a partial integration of the Liouville equation (1.4.21) and standard manipulations. The result is (for $1 \leq j \leq N$):

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k}) f_j^N = \frac{1}{\sqrt{\varepsilon}} T_j^\varepsilon f_j^N + \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon f_{j+1}^N \tag{1.4.24}$$

for $1 \leq j \leq N$. The operator C_{j+1}^ε is defined as:

$$C_{j+1}^\varepsilon = \sum_{k=1}^{j} C_{k,j+1}^\varepsilon \ , \tag{1.4.25}$$

and

$$\begin{aligned} &C_{k,j+1}^\varepsilon f_{j+1}(x_1 \dots x_j; v_1 \dots v_j) = \\ &- \int dx_{j+1} \int dv_{j+1} F\left(\frac{x_k - x_{j+1}}{\varepsilon}\right) \cdot \nabla_{v_k} f_{j+1}(x_1, x_2, \dots, x_{j+1}; v_1, \dots, v_{j+1}). \end{aligned} \tag{1.4.26}$$

$C_{k,j+1}^\varepsilon$ describes the interaction of particle k, belonging to the j-particle subsystem, with a particle outside the subsystem, conventionally denoted by the number $j+1$ (this numbering uses the fact that all the particles are identical). We finally fix the initial value $\{f_j^0\}_{j=1}^N$ of the solution $\{f_j^N(t)\}_{j=1}^N$ assuming that $\{f_j^0\}_{j=1}^N$ is factorized, that is, for all $j = 1, \dots N$

$$f_j^0 = f_0^{\otimes j}, \tag{1.4.27}$$

where f_0 is a given one-particle distribution function. This means that the state of any pair of particles is statistically uncorrelated at time zero. Of course such a statistical independence is destroyed at time $t > 0$ because dynamics creates correlations and eq.n (1.4.24) shows that the time evolution of f_1^N is determined by the knowledge of f_2^N which turns out to be dependent on f_3^N and so on. However, since the interaction between two given particles is going to vanish in the limit $\varepsilon \to 0$, we can hope that such statistical independence is recovered in the same limit[4]. Therefore we expect that when $\varepsilon \to 0$ the one-particle distribution function f_1^N converges to the solution of a suitable nonlinear kinetic equation f, which we are going to investigate. If we expand $f_j^N(t)$ as a perturbation of the free flow $S(t)$ defined as

$$(S(t)f_j)(\mathbf{x}_j, \mathbf{v}_j) = f_j(\mathbf{x}_j - \mathbf{v}_j t, \mathbf{v}_j), \tag{1.4.28}$$

we find

$$\begin{aligned} f_j^N(t) = &S(t) f_j^0 + \frac{N-j}{\sqrt{\varepsilon}} \int_0^t S(t-t_1) C_{j+1}^\varepsilon f_{j+1}^N(t_1) dt_1 + \\ &\frac{1}{\sqrt{\varepsilon}} \int_0^t S(t-t_1) T_j^\varepsilon f_j^N(t_1) dt_1. \end{aligned} \tag{1.4.29}$$

[4]Observe that the physical meaning of propagation of chaos in the weak–coupling contest is different from that arising in the low–density contest. In the weak–coupling regime two particles interact but the effect of the collision is small, while in the low–density case the effect of a collision between two given particles is large, but unlikely.

We now try to keep information on the limit behavior of $f_j^N(t)$. Assuming for the moment that the time evolved j-particle distributions $f_j^N(t)$ are smooth (in the sense that the first and second derivatives are uniformly bounded in ε), then

$$C^{\varepsilon}_{j+1} f^N_{j+1}(\mathbf{x}_j;\mathbf{v}_j;t_1) = \tag{1.4.30}$$
$$-\varepsilon^3 \sum_{k=1}^{j} \int dr \int dv_{j+1} F(r)\cdot\nabla_{v_k} f_{j+1}(\mathbf{x}_j, x_k-\varepsilon r;\mathbf{v}_j, v_{j+1}, t_1).$$

Because of the identity

$$\int dr F(r) = 0, \tag{1.4.31}$$

we find that

$$C^{\varepsilon}_{j+1} f^N_{j+1}(\mathbf{x}_j;\mathbf{v}_j;t_1) = \mathcal{O}(\varepsilon^4) \tag{1.4.32}$$

provided that $D^2_v f^N_{j+1}$ is uniformly bounded. Since

$$\frac{N-j}{\sqrt{\varepsilon}} = \mathcal{O}(\varepsilon^{-\frac{7}{2}})$$

we see that the second term in the right hand side of (1.4.29) does not give any contribution in the limit. Moreover

$$\int_0^t S(t-t_1) T^{\varepsilon}_j f^N_j(t_1) dt_1 = \tag{1.4.33}$$
$$\sum_{i\neq k} \int_0^t dt_1 F\left(\frac{(x_i-x_k)-(v_i-v_k)(t-t_1)}{\varepsilon}\right) \tilde{f}^N_j(\mathbf{x}_j,\mathbf{v}_j;t_1)$$

where $\tilde{f}^N_j$ is a smooth function. We note that the time integral in (1.4.33) is $\mathcal{O}(\varepsilon)$ because $F \neq 0$ only for times in an interval of length $\mathcal{O}(\varepsilon)$. Therefore f^N_j cannot be smooth since we expect a nontrivial limit (for a detailed discussion on this topic we refer to Chapter 3).
Therefore the heuristic idea is to write down the series expansion of the solution, for instance for the one–particle marginal:

$$f_1^N(t) = \sum_{n=0}^{+\infty} \sum_{\Gamma(n)} \alpha(\Gamma(n)) \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_n} [S(t-t_1)O_1 S(t_1-t_2)\ldots O_n S(t_n)] f_n^0\,;$$

where the operator O_j expresses the creation of a new particle (C_j) or a recollision between two particles (T_j), $\Gamma(n)$ is a sequence of indices $\{(r_i,l_i)\}_{i=1}^n$ which represents the particles involved in the interaction at time t_i and such that $r_i < l_i$, $n-1$ is the number of created particles, and the term $\alpha(\Gamma(n))$ is a combinatorial factor. We are not able to analyze the whole series, but we can find, at least formally, an agreement between the particle system and the Landau equation up to the first order in time. We refer to Chapter 3 for a rigorous argument and a detailed discussion.

This thesis contains the following papers: the preprint [PSS] written in collaboration with M. Pulvirenti and S. Simonella, which is presented in Chapter 2; the published paper [BPS] written in collaboration with A.V. Bobylev and M. Pulvirenti, presented in Chapter 3; the published paper [MPS] written in collaboration with E. Miot and M. Pulvirenti, presented in Chapter 4, Sections 4.4–4.6; the preprint [DMS] written in collaboration with L. Desvillettes and E. Miot, presented in Chapter 5, Sections 5.3–5.4.

Chapter 2

Low–density limit

In this Chapter we present [PSS].

2.1 On the validity of the Boltzmann equation for short range potentials

ABSTRACT. We consider a classical system of point particles interacting by means of a short range potential. We prove that, in the low–density (Boltzmann–Grad) limit, the system behaves, for short times, as predicted by the associated Boltzmann equation. This is a revisitation and an extension of the thesis of King [9] (appeared after the well known result of Lanford [10] for hard spheres) and of a recent paper by Gallagher et al [5]. Our analysis applies to any stable and smooth potential. In the case of repulsive potentials (with no attractive parts), we estimate explicitly the rate of convergence.

KEYWORDS. Kinetic Theory, scaling limit, BBGKY hierarchy, Boltzmann equation.

2.2 Introduction

In a well known paper in 1975, O. Lanford presented the first mathematical proof of the validity of the Boltzmann equation for a system of hard spheres, for a sufficiently small time. The starting point was the series expansion describing the time evolution of the statistical states of a hard–sphere system. This series is the solution of a hierarchy of equations formally established by C. Cercignani in 1972 [2], following previous ideas due to H. Grad [6].

The main idea of Lanford is to compare such a series expansion with the one arising from the solution of the Boltzmann equation, claiming the term by term convergence in the so called Boltzmann–Grad limit (BG limit in the sequel). The restriction to short times is due to the fact that the two series have been proven to converge absolutely only for a small time interval. Actually it was remarked in [20] that the Lanford's approach is a Cauchy-Kowalevski kind of argument.

In [10], although all the main ideas, as well as the strategy of the proof, were clearly discussed, the details were missing. The complete proof was presented later on in [9], [16], [19], [18] and [3].

We mention also that the ideas of Lanford can be applied to derive the Boltzmann equation globally in time, in the special case of an expanding cloud of a rare gas in the vacuum [7, 8].

Shortly after the appearance of the Lanford's paper, F. King in his unpublished thesis [9] approached the same validity problem for a particle system interacting by means of a positive, smooth and short range potential. In this case the basic starting point was not the usual BBGKY hierarchy, but a variant of that due to H. Grad [6] (we shall call it the "Grad hierarchy" in the sequel) making the system more similar to a hard–sphere one. More precisely, in [6] only the first equation of this hierarchy was discussed, while the full hierarchy was introduced and derived in [9].

The Boltzmann equation considered by King was written in unusual form. Namely, calling $f = f(x, v, t)$ the distribution function,

$$(\partial_t + v \cdot \nabla_x) f(x, v, t) = \int_{\mathbb{R}^3} dv_1 \int_{S^2_+} d\nu \; (v - v_1) \cdot \nu \Big\{ f(x, v_1', t) f(x, v', t) - f(x, v_1, t) f(x, v, t) \Big\} \tag{2.2.1}$$

where $S^2_+ = \{\nu \in S^2 | \; (v - v_1) \cdot \nu \geq 0\}$, S^2 is the unit sphere in $\mathbb{R}^3$, (v, v_1) is a pair of velocities in incoming collision configuration –see also [1]– and (v', v_1') is the corresponding pair of outgoing velocities defined by

$$\begin{cases} v' = v - \omega[\omega \cdot (v - v_1)] \\ v_1' = v_1 + \omega[\omega \cdot (v - v_1)] \end{cases} . \tag{2.2.2}$$

Here $\omega = \omega(\nu, V)$ is the unit vector bisecting the angle between the incoming relative velocity $V = v_1 - v$ and the outgoing relative velocity $V' = v_1' - v'$ as specified in the figure below.

A more handable and usual form for the Boltzmann equation is obtained by expressing everything in terms of ω, namely

$$(\partial_t + v \cdot \nabla_x) f(x, v, t) = \int_{\mathbb{R}^3} dv_1 \int_{S^2} d\omega \; B(\omega, V) \Big\{ f(x, v_1', t) f(x, v', t) - f(x, v_1, t) f(x, v, t) \Big\} \tag{2.2.3}$$

where $B(\omega, V)/|V|$ is the *differential cross–section* of the potential under consideration with respect to the solid angle ω (in the case of hard spheres, the two formulations (2.2.1) and (2.2.3) coincide since $\omega = \nu$).

After many years, the argument has been recently reconsidered by I. Gallagher et al in a long and self–contained paper [5] pointing out some important facts, surprisingly not discussed in the previous literature. In particular, the term by term convergence is not innocent because B is, in general, not bounded and even not defined as a single–valued function. For instance,

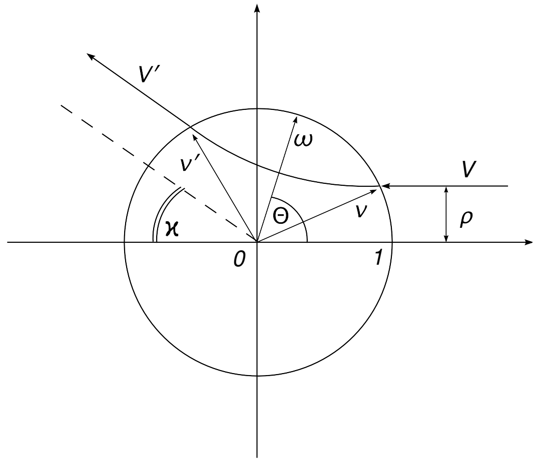

Figure 2.1: The two–body scattering. We denote by ρ the *impact parameter* expressed in microscopic unities ($\rho \in [-1, 1]$) and by $\chi = \chi(\rho, |V|)$ the *scattering angle* ($\chi \in (-\pi, \pi]$, $\chi > 0$ in the figure), while Θ is the angle given by the relation $\chi = \pi - 2\Theta$. We call *scattering vector* the function $\omega = \omega(\nu, V)$.

for smooth positive and bounded potentials (considered by King himself), $\nu \to \omega$ is not globally invertible and B is unbounded. Therefore the difficulty is that one has to exclude concentration of measure on certain small sets in the phase space leading to an evolution much different from the typical Boltzmann behavior. These "bad" events are: (i) the *long time* two–body *scattering*; (ii) the *recollisions*, i.e. the presence of a given pair of particles undergoing two or more collisions. The latter is the main obstacle in proving that the particle system behaves as predicted by the Boltzmann equation.

In [5] the authors prove the validity of the Boltzmann equation under the hypotheses that the potential is well behaving in this sense, namely that the cross–section exists as a single–valued and sufficiently regular function. In the present paper we show that, under very general assumptions on the potential, the Boltzmann equation can indeed be derived in the form (2.2.1). We review the results in [9], completing some parts of the proof and taking care of some inconsistencies. Once the Boltzmann equation has been derived in the form (2.2.1), the passage to the form (2.2.3) is a matter of analysis of the two–body problem. If the cross–section is not a single–valued function, the function B appearing in (2.2.3) can be still expressed as a sum of the contributions arising from each monotonicity branch.

The approach discussed in [5] makes explicit use of the cross–section as a tool for the control of recollisions. In the present paper the aim is to establish a proof that does not depend on any detail of the scattering process. In particular, the term by term convergence (which is the most delicate point in the proof of our main results) is treated in a different way from the one in [5] and [9]: see Section 2.8.1 for a presentation of the problem, and Sections 2.8.2, 2.8 for a quick abstract and an explicit constructive proof respectively. In our method a very useful tool is a *tree expansion* describing the time evolution of the marginals of a statistical state. This is presented in Section 2.7. In Section 2.3 we introduce the mechanical system of particles under examination and make some preliminary remark about it, while in Section 2.4, along the lines of [9], we derive the Grad hierarchy, that is the starting point of our study. In Section 2.5 we fix the hypotheses on the initial data and state our main results. In Section 2.6 we present the uniform short time estimates on the series expansion for the evolution of the marginals. The results in Sections 2.4 and 2.6 are well known by [9] and [5]: we discuss them here briefly for the sake of completeness. Finally, in the Appendix we give sufficient conditions on the interaction for having a bounded or a single–valued cross–section.

One advantage of the methods developed in this paper is that they allow an explicit estimate of the error in the convergence to the Boltzmann equation, as soon as one has explicit estimates of the interaction time of the two–body process in the space of the scattering parameters. Moreover, convergence is established in a strong sense, that is uniformly outside a precise pathological null–measure subset of the phase space.

The analysis of sections 2.3–2.8 can be applied to any smooth and repulsive potential, enlarging the class of interactions considered in [5]. If the potential has an attractive part, there is also an additional difficulty due to long time scattering phenomena and to the presence of trapping orbits in the two–body process. For the sake of clearness we treat this case separately in Section 2.9, where we explain how the proof can be adapted to extend the convergence result, assuming the stability of the interaction.

2.3 The hamiltonian system

We consider a system of N identical classical point particles of unit mass, moving in the whole space and interacting by means of a two–body, short range potential Φ. We denote by $(q_1, v_1, \cdots, q_N, v_N)$ a state of the system, where q_i and v_i indicate the position and the velocity of the particle i, and $q_i(\tau)$ the position of particle i at time τ. The $N-$particle Hamiltonian is

$$H = \frac{1}{2}\sum_{i=1}^{N} v_i^2 + \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \Phi(q_i - q_j) \ . \tag{2.3.1}$$

The dynamical flow is obtained by solving the Newton equations

$$\frac{d^2 q_i}{d\tau^2}(\tau) = \sum_{j\neq i} F(q_i(\tau) - q_j(\tau)) \tag{2.3.2}$$

where $F(q_i - q_j) = F_{i,j} = -\nabla\Phi(q_i - q_j)$ is the force due to the particle j, acting on the particle i. We will assume Φ to be smooth enough in order to have existence and uniqueness of the solution to (2.3.2) for any initial datum such that $q_i \neq q_j$ (see Hypothesis 1 in Section 2.3.2, and Section 2.9).

Consider now a small parameter ε indicating the ratio between the macro and the micro unities. We pass to macroscopic variables defining

$$x = \varepsilon q; \quad t = \varepsilon \tau \ . \tag{2.3.3}$$

In these variables the equations of motion become

$$\frac{d^2 x_i}{dt^2}(t) = \frac{1}{\varepsilon} \sum_{j \neq i} F\left(\frac{x_i(t) - x_j(t)}{\varepsilon}\right) \ . \tag{2.3.4}$$

In order to have a kinetic picture, a tagged particle, say particle 1, must deliver a finite number of collisions in a macroscopic unit time. As a consequence, the density $N\varepsilon^3$ must vanish. More precisely N should be $O(\varepsilon^{-2})$. Indeed, assuming the characteristic interaction length of the potential Φ to be one in microscopic variables, namely $\Phi(q) = 0$ if $|q| > 1$, consider the tube spanned by particle 1 in the (macro) time 1:

$$\left\{x \mid \inf_{0 \leq t \leq 1} |x - x_1(t)| \leq \varepsilon\right\} . \tag{2.3.5}$$

The number of particles in the tube is the number of particles potentially interacting with particle 1 in a macroscopic unit time. Hence, if $N = O(\varepsilon^{-2})$, such a number is expected to be finite. Therefore the scaling we will consider is

$$N \to \infty, \qquad \varepsilon \to 0, \qquad N\varepsilon^2 = l^{-1} > 0, \tag{2.3.6}$$

for a system of N particles obeying to (2.3.4), where $l > 0$ will be proportional to the mean free path and will be fixed to one for notational simplicity.

The scaling (2.3.6) is usually called low–density limit and it is equivalent to the BG limit originally introduced for the hard–sphere system, [6]. If we want to picture the dynamics in macroscopic variables, we can say that a triple collision - namely a situation in which three or more particles are simultaneously interacting - will be very unlikely. Moreover a two–body collision - namely a scattering process involving only two particles - will take place on a scale of time of $O(\varepsilon)$, but since the force is $O(\varepsilon^{-1})$ it will produce a finite effect. In other words the expected dynamics is qualitatively similar to that of the hard–sphere systems.

2.3.1 Statistical description

We want to describe our system from a statistical viewpoint. We will use bold letters for vectors of variables, for instance

$$\mathbf{z}_j = (z_1, \cdots, z_j), \qquad \mathbf{z}_{j,n} = (z_{j+1}, \cdots, z_{j+n}), \qquad z_i = (x_i, v_i) \tag{2.3.7}$$

will be the notation for the state of particles $1, \cdots, j$ and $j+1, \cdots, j+n$ respectively, having position and velocity (x_i, v_i). As usual we introduce the *phase space*

$$\mathcal{M}_N = \left\{\mathbf{z}_N \in \mathbb{R}^{6N} \;\middle|\; |x_i - x_k| > 0, \; i, k = 1 \cdots N, \; k \neq i\right\} . \tag{2.3.8}$$

Consider a probability distribution with density W^N, which is initially (and hence at any positive time) symmetric in the exchange of the particles. Its time evolution is described by the Liouville equation, which reads as

$$(\partial_t + \mathcal{L}_N)W^N = 0 \; , \tag{2.3.9}$$

where the Liouville operator $\mathcal{L}_N$ is

$$\mathcal{L}_N = \mathcal{L}_N^0 + \mathcal{L}_N^I \tag{2.3.10}$$

with:

$$\begin{aligned} \mathcal{L}_N^0 &= \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} \; , \\ \mathcal{L}_N^I &= \frac{1}{\varepsilon} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} F_{i,j} \cdot \nabla_{v_i} \end{aligned} \tag{2.3.11}$$

and $F_{i,j} = -\nabla\Phi\left(\frac{x_i - x_j}{\varepsilon}\right)$.

Remark. For simplicity we shall assume in this subsection, as well as in Section 2.4 below, that W^N is a smooth (say C^1) function of its variables over all $\mathcal{M}_N \times \mathbb{R}^+$, with $v_i \cdot \nabla_{x_i} W^N$, $F_{i,j} \cdot \nabla_{v_i} W^N \in L^1(\mathcal{M}_N)$. The assumption is used to write the Liouville equation as a partial differential equation and to perform partial integrations. This is not really needed to state our results: we will weaken the regularity hypothesis later on by using a density argument (see Proposition 1 and the discussion before it).

We introduce the marginals $g_j^N(\mathbf{z}_j, t)$ of the time evolved measure $W^N(\mathbf{z}_N, t)$, defined by

$$g_j^N(\mathbf{z}_j, t) = \int d\mathbf{z}_{j,N-j} W^N(\mathbf{z}_j, \mathbf{z}_{j,N-j}, t) \; , \tag{2.3.12}$$

which denote the probability distributions of the first j particles (or of any other fixed group of j particles). Clearly $g_N^N = W^N$.

From (2.3.9) and (2.3.12) it follows that the family $\{g_j^N\}_{j=1}^N$ satisfies the well known *BBGKY hierarchy* ([6]):

$$\begin{aligned} &\left(\partial_t + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i}\right) g_j^N + \frac{1}{\varepsilon} \sum_{\substack{i,k=1 \\ i \neq k}}^{j} F\left(\frac{x_i - x_k}{\varepsilon}\right) \cdot \nabla_{v_i} g_j^N \\ &= -\frac{N-j}{\varepsilon} \sum_{i=1}^{j} \int dx_{j+1} \int dv_{j+1} F\left(\frac{x_i - x_{j+1}}{\varepsilon}\right) \cdot \nabla_{v_i} g_{j+1}^N \; . \end{aligned} \tag{2.3.13}$$

Notice that, for a fixed j, the interaction term in the left hand side of Eq. (2.3.13) is, in a sense, negligible because the collisions among a tagged group of particles are unlikely (the potential is indeed vanishing as soon as ε is smaller than $|x_i - x_k|$). Moreover the integral in the right hand side is $O(\varepsilon^3)$. The right hand side, which is due to the interaction between the group of the first j particles with the rest of the system, is $O(1)$ whenever $N = O(\varepsilon^{-2})$, which is exactly the reason why we perform the low–density scaling. However, instead of using the above hierarchy, not very well suited for such a scaling, we will introduce, in Section 2.4, another set of equations.

2.3.2 The two–body scattering

Let us discuss here the scattering process between two particles, which will play a crucial role in what follows. We turn back to microscopic unities, where the potential is assumed to have range one.

Let q_1, v_1, q_2, v_2 be positions and velocities of two particles which are performing a collision. It is well known that this two–body problem can be reduced to a central–motion problem setting the origin in the center of mass:

$$\frac{q_1 + q_2}{2} = 0, \qquad q = q_1 - q_2 \ . \tag{2.3.14}$$

Then the evolution is given by

$$\frac{d^2 q}{d\tau^2}(\tau) = 2F\left(q(\tau)\right) \ . \tag{2.3.15}$$

The above equation of motion is "almost" explicitly solvable. In particular, fixed the relative velocity $V = v_1 - v_2$ (hence fixed a value of the energy in the center of mass), one can restrict his attention to the control of the scattering function $\omega = \omega(\nu)$. Since the scattering takes place in a plane, this amounts to control the function $\Theta = \Theta(\rho)$ (see Fig. 2.1 in Section 2.2). The classical integral formula expressing Θ in terms of the modulus of the incoming relative velocity $|V|$, the potential Φ and the impact parameter ρ will be written in the Appendix (see Eq. (2.11.2)). That formula is not so easy to handle with, so it will not be employed in the present section.

In what follows it will be rather crucial to have an estimate on the *scattering time* τ_*, namely the measure of the time interval for which $|q(\tau)| < 1$. To this purpose, we need to state our precise assumptions on the potential.

Hypothesis 1. The two–body potential $\Phi = \Phi(q), q \in \mathbb{R}^3$, is radial, with support $|q| < 1$ and not increasing in $|q|$. We assume either $\Phi \in C^2(\mathbb{R}^3)$, or $\Phi \in C^2(\mathbb{R}^3 \setminus \{0\})$ and $\Phi(q) \to +\infty$ as $q \to 0$.

The smoothness assumption is needed to ensure existence and uniqueness of the flow evolution for the system of N particles, while the monotonicity is introduced to allow a simple control on the scattering time τ_*. We defer more general cases to Section 2.9.

We will use sometimes the notational inconsistency for which $\Phi(r) = \Phi|_{|q|=r}$.

Consider the central motion given by Eq. (2.3.15) with the initial conditions describing the two particles just before the interaction, namely $q(0) = \nu \in S^2$, $\dot{q}(0) = V$ and $|V| > 0$, $V \cdot \nu \leq 0$. Denote

$$L = |\nu \wedge V| = |\rho V| \in [0, V] \tag{2.3.16}$$

the magnitude of angular momentum, being ρ the impact parameter (Fig. 2.1). A rather general estimate on τ_* is the following:

Lemma 1. *Under Hypothesis 1 it is*

$$\tau_* \leq \frac{A}{L} \tag{2.3.17}$$

for some $A > 0$.

Proof. From the conservation laws one derives the well known formula expressing τ_* as a function of V and L :

$$\tau_* = \sqrt{2} \int_{r_*}^{1} dr \frac{1}{\left(\frac{V^2}{2} - \frac{L^2}{2r^2} - 2\Phi(r)\right)^{1/2}} \ , \tag{2.3.18}$$

where r_* is the minimum distance from the origin, $r_* = \inf_{\tau \in (0,\tau_*)} |q(\tau)|$, related to V and L by

$$\frac{V^2}{2} = \frac{L^2}{2r_*^2} + 2\Phi(r_*) \ . \tag{2.3.19}$$

The effective potential, i.e. the potential of the reduced one–dimensional motion (which is the evolution of the radial coordinate in the system of the center of mass), is the $L-$dependent function

$$2\Phi_{eff}(r) = \frac{L^2}{2r^2} + 2\Phi(r) - \frac{L^2}{2} \ , \qquad r \in [0, 1] \ . \tag{2.3.20}$$

We can write

$$\begin{aligned}\tau_* &= \int_{r_*}^{1} dr \frac{1}{(\Phi_{eff}(r_*) - \Phi_{eff}(r))^{1/2}} \\ &\leq \frac{1}{\left(\min_{[0,1]}(-\Phi'_{eff})\right)^{\frac{1}{2}}} \int_{r_*}^{1} dr \frac{1}{\sqrt{r - r_*}} \ .\end{aligned} \tag{2.3.21}$$

Denote improperly Φ' the derivative with respect to r of the function $\Phi|_{|q|=r}$. Since $\Phi' \leq 0$ and

$$\Phi'_{eff}(r) = \Phi'(r) - \frac{L^2}{2r^3} \ , \tag{2.3.22}$$

the result follows easily. □

The estimate in Lemma 1 tells us that $\tau_* = O((\rho V)^{-1})$. This has the advantage to be general and sufficient to our purposes. Clearly the bound can be improved in many cases. Singularities in the scattering occur whenever the collision is central ($\rho = 0$) and the energy corresponds exactly to a point of vanishing force ($V^2/4 = \Phi(r), \Phi'(r) = 0$). This kind of singularities does not exist if the potential is unbounded at the origin and strictly repulsive: for instance for potentials diverging at the origin with a power law, formula (2.3.17) can be easily replaced by $\tau_* = O(V^{-1})$. From (2.3.21) it can be noticed also that the singularity for low energies may appear only if Φ goes to zero smoothly (C^1) in $r = 1$.

We conclude by introducing a map which encodes all the properties of the two–body interaction. The *scattering operator* $\mathcal{I}$ is defined over

$$\Big\{(\nu, V) \in S^2 \times \mathbb{R}^3 \setminus \{0\} \text{ s.t. } V \cdot \nu \leq 0\Big\} \tag{2.3.23}$$

by:

$$\begin{aligned} &\mathcal{I}(\nu, V) = (\nu', V') \\ &\begin{cases} V' = V - 2\omega(\omega \cdot V) \\ \nu' = -\nu + 2\omega(\omega \cdot \nu) \end{cases} \end{aligned} \tag{2.3.24}$$

where $\omega = \omega(\nu, V)$ is the scattering vector, see Fig. 2.1. It follows that $V \cdot \nu = -V' \cdot \nu'$. In particular, $V' \cdot \nu' \geq 0$, i.e. $\mathcal{I}$ sends incoming to outgoing configurations.

The following property of $\mathcal{I}$ will be used in the validation of the Boltzmann equation.

Lemma 2. *$\mathcal{I}$ is an invertible transformation that preserves Lebesgue measure.*

Proof. Of course, being the dynamics reversible, $\omega(\nu', V') = \omega(\nu, V)$ (see Fig. 2.1) and $\mathcal{I}^{-1}$ is defined in the same way as $\mathcal{I}$.

To see that $\mathcal{I}$ is measure preserving, we fix cartesian coordinates on the plane where the scattering occurs, and call ϕ the angle formed by V and the first axis (with ϕ growing when V rotates counterclockwise), α the angle formed by V and ν and such that $\sin\alpha$ is the impact parameter ρ (with the convention $\alpha \in [\pi/2, (3\pi)/2]$, see Fig. 2.1). Restricting to the plane of the scattering, we have the parametrization $V = (|V|\cos\phi, |V|\sin\phi)$, $\nu = \alpha$. In the variables $|V|, \phi, \alpha$ the action of $\mathcal{I}$ is simply described by:

$$V' = (|V'|\cos\phi', |V'|\sin\phi')\ , \qquad \nu' = \alpha'\ ,$$

where

$$\begin{cases} |V'| = |V| \\ \alpha' = \pi - \alpha \\ \phi' = \phi - \chi(\sin\alpha, |V|) \end{cases} . \tag{2.3.25}$$

Note that $\alpha' \in [-\pi/2, \pi/2]$. The first equation is conservation of energy, the second conservation of angular momentum, and the third holds by definition of scattering angle (Fig.2.1). It can be shown that χ is a differentiable function of its arguments (see the discussion in the Appendix). Moreover, the determinant of the jacobian of the transformation (2.3.25) has modulus one, independently of the form of χ. This concludes the proof. □

2.4 The Grad hierarchy

In this section we derive a hierarchy of equations for a family of quantities which are very close to the marginals introduced in the previous Section 2.3.1. This allows to put the dynamical problem in a form somehow similar to the one arising in considering hard–sphere systems and more suitable for the study of the low–density limit.

We define the *reduced marginals*

$$f_j^N(\mathbf{z}_j, t) = \int_{S(\mathbf{x}_j)^{N-j}} d\mathbf{z}_{j,N-j} W^N(\mathbf{z}_j, \mathbf{z}_{j,N-j}, t) \ , \tag{2.4.1}$$

where

$$S(\mathbf{x}_j) = \left\{ z = (x, v) \in \mathbb{R}^6 \ \middle| \ |x - x_k| > \varepsilon \ \text{ for all } k = 1, \cdots, j \right\} . \tag{2.4.2}$$

It is clear that the functions f_j^N, for any j, are asymptotically equivalent (uniformly on compact sets in $\mathcal{M}_j$ in the BG limit) to the usual marginals.

Consider now a configuration $\mathbf{z}_N = (\mathbf{z}_j, \mathbf{z}_{j,N-j})$ such that

$$|x_\ell - x_k| > \varepsilon \tag{2.4.3}$$

for all $\ell = 1, \cdots, j$ and $k = j+1, \cdots, N$. Since the range of the interaction is ε, the interaction between the group of the first j particles and the rest of the system is vanishing. Therefore the Liouville equation (2.3.9) on such configurations becomes:

$$\partial_t W^N + \mathcal{L}_N^0 W^N + \mathcal{L}_j^I W^N + \mathcal{L}_{j,N}^I W^N = 0 \ , \tag{2.4.4}$$

where $\mathcal{L}_{j,N}^I$ is defined as in (2.3.11) with the sums running from $j+1$ to N.

As already said in the Remark at page 28, we make at the moment the regularity assumptions needed to justify Eq. (2.4.4) and all the steps in the following derivation (see the mentioned Remark).

Integrating Eq. (2.4.4) with respect to $d\mathbf{z}_{j,N-j}$ over $S(\mathbf{x}_j)^{N-j}$ we obtain:

$$\begin{aligned} \left(\partial_t + \mathcal{L}_j^I\right) f_j^N(\mathbf{z}_j, t) = &- \sum_{i=j+1}^{N} \int_{S(\mathbf{x}_j)^{N-j}} d\mathbf{z}_{j,N-j} v_i \cdot \nabla_{x_i} W^N(\mathbf{z}_N, t) \\ &- \sum_{i=1}^{j} \int_{S(\mathbf{x}_j)^{N-j}} d\mathbf{z}_{j,N-j} v_i \cdot \nabla_{x_i} W^N(\mathbf{z}_N, t) \ , \end{aligned} \tag{2.4.5}$$

where we used that

$$\int_{\mathbb{R}^{3(N-j)}} d\mathbf{v}_{j,N-j} \mathcal{L}_{j,N}^I W^N = 0 \ . \tag{2.4.6}$$

The first sum is handled by the divergence theorem yielding

$$- \sum_{i=j+1}^{N} \int_{S(\mathbf{x}_j)} dz_{j+1} \cdots \int_{\partial S(\mathbf{x}_j)} d\sigma(x_i) dv_i \cdots \int_{S(\mathbf{x}_j)} dz_N (v_i \cdot \nu_i) W^N \ , \tag{2.4.7}$$

where ν_i is the outward normal to $S(\mathbf{x}_j)$ and $d\sigma(x_i)dv_i$ is the surface measure. Note that if $z_k \in \partial S(\mathbf{x}_j)$, there exists an index $k \in \{1, \cdots, j\}$ such that $|x_k - x_i| = \varepsilon$. Moreover, if $\mathbf{z}_j \in \mathcal{M}_j$, there is only one such an index for almost all x_i with respect to the surface measure $d\sigma(x_i)$. Hence $\partial S(\mathbf{x}_j)$, as regards the x-dependence, is the disjoint union of pieces of spherical surfaces. We call such pieces $\sigma_k(\mathbf{x}_j)$, that is

$$\partial S(\mathbf{x}_j) = \bigcup_{k=1}^{j} \sigma_k(\mathbf{x}_j) \times \mathbb{R}^3 \tag{2.4.8}$$

where

$$\sigma_k(\mathbf{x}_j) \subset \{x \mid |x - x_i| = \varepsilon\} \ . \tag{2.4.9}$$

We set $\nu_{k,i} = \frac{x_k - x_i}{|x_k - x_i|}$. Using the symmetry of W^N we have $N-j$ identical integrals, for which (2.4.7) becomes:

$$-(N-j)\sum_{k=1}^{j}\int_{\sigma_k(\mathbf{x}_j)} d\sigma(x_{j+1})\int_{\mathbb{R}^3} dv_{j+1}(v_{j+1}\cdot\nu_{k,j+1})\int_{S(\mathbf{x}_j)^{N-j-1}} d\mathbf{z}_{j+1,N-j-1}W^N \ . \tag{2.4.10}$$

The integration domain of the last integral in (2.4.10) is not $S(\mathbf{x}_{j+1})^{N-j-1}$, as it would be necessary to recover f_{j+1}^N and close the equation. We could reduce this integration to $S(\mathbf{x}_{j+1})^{N-j-1}$, and this would produce a small error in the BG limit. Nevertheless, we want to establish an exact and closed equation, for which we need a further work. Let $\boldsymbol{i} = \{i_1, \cdots, i_m\}$ be a subset of ordered indices of $\{j+2, \cdots, N\}$, with $i_1 < i_2 < \cdots < i_m$. We put $\mathbf{z}_{\boldsymbol{i}} = \{z_{i_1}, \cdots, z_{i_m}\}$, and introduce the set

$$\Delta_{\boldsymbol{i}}(\mathbf{x}_{j+1}) := \Big\{\mathbf{z}_{\boldsymbol{i}} \subset S(\mathbf{x}_j)^m \text{ such that, for each } \ell = 1, \cdots, m, \ \min_{\substack{k\in \boldsymbol{i}\cup\{j+1\}\\ k\neq i_\ell}} |x_k - x_{i_\ell}| \leq \varepsilon\Big\} \ . \tag{2.4.11}$$

A generic configuration in $S(\mathbf{x}_j)^{N-j-1}$ differs from $S(\mathbf{x}_{j+1})^{N-j-1}$ because some particle, say particle i_1, could *overlap* with particle $j+1$, this meaning that $|x_{i_1} - x_{j+1}| \leq \varepsilon$. If this is the case, we consider the maximal *cluster* of overlapping particles with indices $\boldsymbol{i}$. Definition (2.4.11) gives the subset of such cluster–configurations. The other particles are far apart the group with indices $1, \cdots, j, j+1, \boldsymbol{i}$, therefore each of them is in $S(\mathbf{x}_{j+1}, \mathbf{x}_{\boldsymbol{i}})$. Then, we can decompose the integration domain $S(\mathbf{x}_j)^{N-j-1}$ in (2.4.10) as a union of disjoint sets to obtain

$$\begin{aligned}-(N-j)\sum_{k=1}^{j}\sum_{m=0}^{N-j-1}\sum_{\boldsymbol{i}:|\boldsymbol{i}|=m}\int_{\sigma_k(\mathbf{x}_j)} d\sigma(x_{j+1})\int_{\mathbb{R}^3} dv_{j+1}(v_{j+1}\cdot\nu_{k,j+1})&\\ \cdot\int_{\Delta_{\boldsymbol{i}}(\mathbf{x}_{j+1})} d\mathbf{z}_{\boldsymbol{i}} f_{j+1+m}^N(\mathbf{z}_{j+1}, \mathbf{z}_{\boldsymbol{i}}) \ ,&\end{aligned} \tag{2.4.12}$$

where we denoted $|\boldsymbol{i}|$ the cardinality of $\boldsymbol{i}$ and we used (2.4.1), as well as the symmetry of W^N, to integrate out the not–clusterized variables. All the terms in the $\sum_{\boldsymbol{i}}$ are identical so that the result can be written

$$-\sum_{k=1}^{j}\sum_{m=0}^{N-j-1}(N-j)(N-j-1)\cdots(N-j-m)\int_{\sigma_k(\mathbf{x}_j)}d\sigma(x_{j+1})$$
$$\cdot\int_{\mathbb{R}^3}dv_{j+1}(v_{j+1}\cdot\nu_{k,j+1})\int_{\Delta_m(\mathbf{x}_{j+1})}\frac{d\mathbf{z}_{j+1,m}}{m!}f^N_{j+1+m}(\mathbf{z}_{j+1+m})\ , \qquad (2.4.13)$$

where

$$\Delta_m(\mathbf{x}_{j+1}) := \Big\{\mathbf{z}_{j+1,m}\subset S(\mathbf{x}_j)^m \text{ such that, for each } \ell=j+2,\cdots,j+1+m,$$
$$\min_{\substack{i\in\{j+1,\cdots,j+1+m\}\\ i\neq\ell}}|x_i-x_\ell|\le\varepsilon\Big\}\ . \qquad (2.4.14)$$

The second sum in Eq. (2.4.5) is

$$-\sum_{i=1}^{j}\int_{S(\mathbf{x}_j)^{N-j}}d\mathbf{z}_{j,N-j}v_i\cdot\nabla_{x_i}W^N(\mathbf{z}_N,t) = -\sum_{i=1}^{j}v_i\cdot\nabla_{x_i}f^N_j(\mathbf{z}_j,t)$$
$$+(N-j)\sum_{i=1}^{j}\int_{\sigma_i(\mathbf{x}_j)}d\sigma(x_{j+1})\int_{\mathbb{R}^3}dv_{j+1}(v_i\cdot\nu_{i,j+1})\int_{S(\mathbf{x}_j)^{N-j-1}}d\mathbf{z}_{j+1,N-j-1}W^N(\mathbf{z}_N,t)\ .$$
$$(2.4.15)$$

Repeating the same step we did before to go from (2.4.10) to (2.4.13), we readily arrive to the following hierarchy of equations (which we call *Grad hierarchy*):

$$(\partial_t+\mathcal{L}_j)\,f^N_j = \sum_{m=0}^{N-j-1}\mathcal{A}^\varepsilon_{j+1+m}f^N_{j+1+m}, \qquad 1\le j\le N\ , \qquad (2.4.16)$$

where the operator $\mathcal{L}_j=\mathcal{L}^\varepsilon_j$ depends also on ε through its interacting part (2.3.11), and

$$\mathcal{A}^\varepsilon_{j+1+m}f^N_{j+1+m}(\mathbf{z}_j,t) = (N-j)(N-j-1)\cdots(N-j-m)$$
$$\cdot\sum_{i=1}^{j}\varepsilon^2\int_{S^2}d\nu\mathbb{1}_{\{\min_{\ell=1,\cdots,j,\ell\neq i}|x_i+\nu\varepsilon-x_\ell|>\varepsilon\}}(\nu)\int_{\mathbb{R}^3}dv_{j+1}(v_{j+1}-v_i)\cdot\nu$$
$$\cdot\int_{\Delta_m(\mathbf{x}_{j+1})}\frac{d\mathbf{z}_{j+1,m}}{m!}f^N_{j+1+m}(\mathbf{z}_j,x_i+\nu\varepsilon,v_{j+1},\mathbf{z}_{j+1,m}t)\ , \qquad (2.4.17)$$

with $x_{j+1}=x_i+\nu\varepsilon$ in the argument of Δ_m. We indicate with $\mathbb{1}_{\{\cdot\}}(\cdot)$ the characteristic function of the set defined by the condition in the curly brackets.

In particular it is

$$\mathcal{A}^\varepsilon_{j+1}f^N_{j+1}(\mathbf{z}_j,t)$$
$$=\varepsilon^2(N-j)\sum_{i=1}^{j}\int_{S^2\times\mathbb{R}^3}d\nu dv_{j+1}\mathbb{1}_{\{\min_{\ell=1,\cdots,j;\ell\neq i}|x_i+\nu\varepsilon-x_\ell|>\varepsilon\}}(\nu)$$
$$\cdot(v_{j+1}-v_i)\cdot\nu f^N_{j+1}(\mathbf{z}_j,x_i+\nu\varepsilon,v_{j+1},t)\ ,$$
$$=\varepsilon^2(N-j)\mathcal{C}^\varepsilon_{j+1}f^N_{j+1}(\mathbf{z}_j,t)\ , \qquad (2.4.18)$$

where (2.4.18) defines $\mathcal{C}^{\varepsilon}_{j+1}$, which is the same collision operator appearing in the hard–sphere case, see [10]. Actually it is clear that, in the BG limit, this term is the only $O(1)$ term in the sum in the right hand side of (2.4.16). Indeed, for $m > 0$ and fixed j, the size of $\mathcal{A}^{\varepsilon}_{j+1+m}$ will be

$$O(N^{m+1}\varepsilon^2\varepsilon^{3m}) \,, \tag{2.4.19}$$

the ε^{3m} coming from the successive integrations in the domain $\Delta_m(\mathbf{x}_{j+1})$. This means that we are in a situation quite similar to that of the hard–sphere system [10], and we can hope to derive the Boltzmann equation in a similar manner.

2.4.1 Series solution

Consider the dynamical flow obtained by solving the Newton equations (2.3.4) for a system of j particles:

$$\frac{d^2 x_i}{dt^2}(t) = \frac{1}{\varepsilon}\sum_{k\neq i} F\left(\frac{x_i(t)-x_k(t)}{\varepsilon}\right) \,, \tag{2.4.20}$$

where i and k run now from 1 to j. Denote by $\mathtt{T}^{\varepsilon}_j(t)\mathbf{z}_j$ the solution of this system of equations with initial datum $\mathbf{z}_j$. The action of this flow on the functions is given by the *interacting flow operator* $\mathcal{S}^{\varepsilon}_j(t)$, defined as

$$\mathcal{S}^{\varepsilon}_j(t)g(\mathbf{z}_j) = g(\mathtt{T}^{\varepsilon}_j(-t)\mathbf{z}_j) \,. \tag{2.4.21}$$

We may represent the solution of Eq (2.4.16) by means of a perturbative expansion, that is just the iteration of the Duhamel formula:

$$\begin{aligned} f^N_j(t) = \sum_{n=0}^{N-j} \sum_{\substack{m_1,\cdots,m_n\geq 0:\\ j+n+\sum_{i=1}^n m_i\leq N}} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ \cdot\mathcal{S}^{\varepsilon}_j(t-t_1)\mathcal{A}^{\varepsilon}_{j+1+m_1}\mathcal{S}^{\varepsilon}_{j+1+m_1}(t_1-t_2)\cdots\mathcal{A}^{\varepsilon}_{j+n+\sum_{i=1}^n m_i}\mathcal{S}^{\varepsilon}_{j+n+\sum_{i=1}^n m_i}(t_n) f^N_{j+n+\sum_{i=1}^n m_i}(0) \,, \end{aligned} \tag{2.4.22}$$

where $f^N_j(t) = f^N_j(\cdot,t)$, and $f^N_j(0)$ are the reduced marginals of the initial probability distribution. This expansion will be our main tool.

We derived Eq. (2.4.22) assuming sufficient smoothness of the initial distribution (see Remark on page 28). However, by using a density approximation, (2.4.22) can be proven to hold for a general class of initial measures. The argument can be found in [19] page 281, or [17] page 18 for cases of hard–sphere dynamics, and it can be applied also to general smooth potentials. We list the main steps in what follows. A different approach based on a weak formulation may be found in [5].

Consider the collection of integration variables in the right hand side of (2.4.22), which we call $\boldsymbol{\lambda} = (t_1,\cdots,t_n,\nu_1,\cdots,\nu_n,v_{j+1},\cdots,v_{j+n},\mathbf{z}_{j+n+\sum_i m_i})$, see also (2.4.17). The reduced marginal in the integrand takes a form $f^N_{j+n+\sum_{i=1}^n m_i}(\mathbf{y}_{j+n+\sum_{i=1}^n m_i}(\mathbf{z}_j,\boldsymbol{\lambda}),0)$, with $\mathbf{z}_j\in\mathcal{M}_j$ and $\boldsymbol{\lambda}$ in the integration domain (for the understanding of the detailed structure of the map, $(\mathbf{z}_j,\boldsymbol{\lambda})\to\mathbf{y}_{j+n+\sum_{i=1}^n m_i}$, we defer to the discussion in Section 2.7 of this paper). Now, let us write the expansion in (2.4.22) for a generic measurable probability density W^N. To have a nice control on the integration over large velocities, we shall assume the exponential decrease $f_j^N \le c^j e^{-\beta \sum_{i=1}^j v_i^2}$ for some $c, \beta > 0$. Since $\mathbf{y}$ is a Borel map (as follows directly from measurability of the partial mappings $(\mathbf{z}_j, t) \to \mathtt{T}_j^\varepsilon(t)\mathbf{z}_j)$, the expansion makes sense for the reduced marginals of W^N, and the integrals therein are absolutely convergent.

To recover identity (2.4.22), we use that there exists a sequence of smooth densities $W^{N,\gamma}$ which evolve according to (2.4.22), satisfy the exponential bound, and approximate W^N : $\lim_{\gamma\to 0} W^{N,\gamma} = W^N$ a.e. on $\mathcal{M}_N$. Since the densities evolve according to $W^N(\mathbf{z}_N, t) = W^N(\mathtt{T}_N^\varepsilon(-t)\mathbf{z}_N)$ (and same equation for $W^{N,\gamma}$), we also have $\lim_{\gamma\to 0} W^{N,\gamma}(t) = W^N(t)$ a.e. on $\mathcal{M}_N$ and, consequently, $\lim_{\gamma\to 0} f_j^{N,\gamma}(t) = f_j^N(t)$ a.e. on $\mathcal{M}_j$. We are left with the problem of taking the limit of the right hand side of (2.4.22). Using the measure preserving property of the flows $\mathtt{T}_j^\varepsilon(t)$, it can be shown that $\mathbf{y}$ is not singular, in the sense that $\mathbf{y}^{-1}$ maps null sets in $\mathcal{M}_{j+n+\sum_{i=1}^n m_i}$ to null sets of values of $(\mathbf{z}_j, \boldsymbol{\lambda})$ in its domain. This fact, together with the gaussian estimate, allows to apply dominated convergence, thus concluding the proof.

Summarizing, we have the following

Proposition 1. *Let W^N be the density of a probability distribution on $\mathcal{M}_N$ with reduced marginals f_j^N. Suppose that W^N is Borel measurable, symmetric in the particle labels and such that $f_j^N \le c^j e^{-\beta\sum_{i=1}^j v_i^2}$ for some $c, \beta > 0$. Then the evolved measure at time $t > 0$ has reduced marginals $f_j^N(t)$ given by Eq.* (2.4.22)*, for almost all points in $\mathcal{M}_j$.*

Remark. It is important to observe that definitions of the operators $\mathcal{A}_j^\varepsilon$ and $\mathcal{C}_j^\varepsilon$ (respectively (2.4.17) and (2.4.18)) involve a trace problem, so that they are well posed if they act over functions which are at least continuous over a.a. points of the spheres of center x_i and radius ε (see definition (2.4.17)). Nevertheless, this is not relevant to our purposes, since we will work only with operators of the form $\int ds \mathcal{A}_j^\varepsilon \mathcal{S}_j^\varepsilon(s)$. These last are indeed well defined over functions f_j^N satisfying the hypotheses of Proposition 1, by virtue of the nonsingularity of the map $\mathbf{y}_j$.

For future convenience, let us conclude this subsection by giving some more definition. The subseries associated to the dominant term of (2.4.22) (that with all $m_i = 0$) defines a new sequence of functions which we call $\{\tilde{f}_j^N\}_{j=1}^N$:

$$\tilde{f}_j^N(t) = \sum_{n=0}^{N-j} \alpha_n^\varepsilon(j) \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \tag{2.4.23}$$

$$\cdot \mathcal{S}_j^\varepsilon(t-t_1)\mathcal{C}_{j+1}^\varepsilon \mathcal{S}_{j+1}^\varepsilon(t_1-t_2)\cdots \mathcal{C}_{j+n}^\varepsilon \mathcal{S}_{j+n}^\varepsilon(t_n) f_{j+n}^N(0) \ ,$$

$$\alpha_n^\varepsilon(j) := \varepsilon^{2n}(N-j)(N-j-1)\cdots(N-j-n+1) \ , \tag{2.4.24}$$

where we used definition (2.4.18). Notice that in the BG limit $\alpha_n^\varepsilon(j) = O(1)$.

Finally, it will be convenient to decompose the collision operator $\mathcal{C}^{\varepsilon}_{j+1}$ in the following form:

$$
\begin{aligned}
&\mathcal{C}^{\varepsilon}_{j+1} = \sum_{k=1}^{j} \mathcal{C}^{\varepsilon}_{k,j+1} \\
&\mathcal{C}^{\varepsilon}_{k,j+1} = \mathcal{C}^{\varepsilon,+}_{k,j+1} - \mathcal{C}^{\varepsilon,-}_{k,j+1} \\
&\mathcal{C}^{\varepsilon,+}_{k,j+1} f^N_{j+1}(\mathbf{z}_j, t) = \int_{S^2_- \times \mathbb{R}^3} d\nu dv_{j+1} \mathbb{1}_{\{\min_{\ell=1,\cdots,j;\ell\neq k} |x_k + \nu\varepsilon - x_\ell| > \varepsilon\}}(\nu) \\
&\qquad\qquad\qquad\qquad \cdot |(v_k - v_{j+1}) \cdot \nu| f^N_{j+1}(\mathbf{z}_j, x_k + \nu\varepsilon, v_{j+1}, t) \\
&\mathcal{C}^{\varepsilon,-}_{k,j+1} f^N_{j+1}(\mathbf{z}_j, t) = \int_{S^2_+ \times \mathbb{R}^3} d\nu dv_{j+1} \mathbb{1}_{\{\min_{\ell=1,\cdots,j;\ell\neq k} |x_k + \nu\varepsilon - x_\ell| > \varepsilon\}}(\nu) \\
&\qquad\qquad\qquad\qquad \cdot |(v_k - v_{j+1}) \cdot \nu| f^N_{j+1}(\mathbf{z}_j, x_k + \nu\varepsilon, v_{j+1}, t)
\end{aligned} \tag{2.4.25}
$$

where

$$
\begin{aligned}
S^2_+ &= \{\nu \mid (v_k - v_{j+1}) \cdot \nu \geq 0\} \ , \\
S^2_- &= \{\nu \mid (v_k - v_{j+1}) \cdot \nu \leq 0\} \ .
\end{aligned} \tag{2.4.26}
$$

2.4.2 The Boltzmann hierarchy

In this subsection we treat formally the solution to the Boltzmann equation (2.2.1) as we did in Section 2.4.1 for the interacting system of particles and compare heuristically the results.

Let f be a solution to Eq. (2.2.1). Consider the products

$$
f_j(\mathbf{z}_j, t) = f(t)^{\otimes j}(\mathbf{z}_j) = f(z_1, t) f(z_2, t) \cdots f(z_j, t) \ . \tag{2.4.27}
$$

It is easy to show that the f_j solve the hierarchy of equations

$$
\left(\partial_t + \mathcal{L}^0_j\right) f_j = \mathcal{C}_{j+1} f_{j+1}, \qquad 1 \leq j < \infty \ , \tag{2.4.28}
$$

where

$$
\mathcal{C}_{j+1} = \sum_{k=1}^{j} \mathcal{C}_{k,j+1} \tag{2.4.29}
$$

$$
\begin{aligned}
&\mathcal{C}_{k,j+1} = \mathcal{C}^+_{k,j+1} - \mathcal{C}^-_{k,j+1} \\
&\mathcal{C}^+_{k,j+1} f_{j+1}(\mathbf{z}_j, t) = \int_{S^2_+ \times \mathbb{R}^3} d\nu dv_{j+1} (v_k - v_{j+1}) \cdot \nu f_{j+1}(z_1, \cdots, x_k, v'_k, \cdots, z_j, x_k, v'_{j+1}, t) \\
&\mathcal{C}^-_{k,j+1} f_{j+1}(\mathbf{z}_j, t) = \int_{S^2_+ \times \mathbb{R}^3} d\nu dv_{j+1} (v_k - v_{j+1}) \cdot \nu f_{j+1}(z_1, \cdots, x_k, v_k, \cdots, z_j, x_k, v_{j+1}, t)
\end{aligned}
$$

and

$$
\begin{cases}
v'_k = v_{j+1} - \omega[\omega \cdot (v_k - v_{j+1})] \\
v'_{j+1} = v_1 + \omega[\omega \cdot (v_k - v_{j+1})]
\end{cases} , \tag{2.4.30}
$$

$\omega = \omega(\nu, v_{j+1} - v_k)$ being the scattering vector (see Fig. 2.1).

The infinite hierarchy of equations (2.4.28) (which does not express nothing else than the Boltzmann equation whenever the factorization property (2.4.27) holds) is called the *Boltzmann hierarchy.* Proceeding as before, we may represent its solution by the perturbative expansion around the free flow:

$$f_j(t) = \sum_{n\geq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ \cdot \mathcal{S}_j(t-t_1)\mathcal{C}_{j+1}\mathcal{S}_{j+1}(t_1-t_2)\cdots\mathcal{C}_{j+n}\mathcal{S}_{j+n}(t_n)f_{j+n}(0) \ , \qquad (2.4.31)$$

where now $\mathcal{S}_j(t)$ is the *free flow operator*, defined as

$$\mathcal{S}_j(t)g(\mathbf{z}_j) = g(x_1 - v_1 t, v_1, \cdots, x_j - v_j t, v_j) \ . \qquad (2.4.32)$$

Note that:

- Eq. (2.4.22) is an identity which expresses f_j^N (well defined by means of the N–particle flow) in terms of a finite sum of operators acting on the initial sequence $f_j^N(0)$;
- $\tilde{f}_j^N$, Eq. (2.4.23), is just a technical definition;
- Eq. (2.4.31) is a series whose convergence must be proven.

As for the hard–sphere case [10], it is possible to show that such a series is indeed convergent for a short time. We will show it in Section 2.6.

Remark. This last result implies also local existence and uniqueness of the solution to the time–integrated version of the Boltzmann hierarchy in the class of continuous functions such that $f_j(t) \leq c^j e^{-\beta\sum_{i=1}^j v_i^2}$ for some $c, \beta > 0$ (see e.g. [3]). In particular, in the case of initial product states, factorization is propagated in time and each factor is the local solution of the time–integrated Boltzmann equation.

Reminding the discussion at the end of Section 2.4 and the fact that $\mathcal{L}_j^I$ equals zero for ε small, we shall guess that (2.4.28) is what one gets just letting ε go to zero in the Grad hierarchy (2.4.16). To do so, assume for simplicity that f_{j+1}^N is continuous. Then, we may try to rewrite the action of $\mathcal{C}_{k,j+1}^{\varepsilon,+}$ on f_{j+1}^N in such a way that the function is evaluated in *incoming* collision states. Call $t_* = \varepsilon\tau_*$ the time of interaction of particles k and $j+1$. Since the scattering process is, in macroscopic variables, almost instantaneous, we assume that the other particles do not interact in the same time interval. By the continuity of the flow it will be

$$\mathtt{T}_{j+1}^{\varepsilon}(-t_*)(\mathbf{z}_j, x_k + \nu\varepsilon, v_{j+1}) \approx (z_1, \cdots, x_k, v_k', \cdots, z_j, x_k, v_{j+1}') \ , \qquad (2.4.33)$$

hence

$$\begin{aligned} &\mathcal{C}_{k,j+1}^{\varepsilon,+} f_{j+1}^N(\mathbf{z}_j, t) \qquad (2.4.34) \\ &\quad \approx \int_{S_-^2\times\mathbb{R}^3} d\nu dv_{j+1} |(v_k - v_{j+1})\cdot\nu| f_{j+1}^N(z_1, \cdots, x_k, v_k', \cdots, z_j, x_k, v_{j+1}', t) \\ &\quad = \int_{S_+^2\times\mathbb{R}^3} d\nu dv_{j+1} (v_k - v_{j+1})\cdot\nu f_{j+1}^N(z_1, \cdots, x_k, v_k', \cdots, z_j, x_k, v_{j+1}', t) \ , \end{aligned}$$

where in the second step we simply changed $\nu \to -\nu$.

We stress that this heuristic discussion is somehow dangerous. In fact, the required continuity property of f^N_{j+1}, even when true for any fixed N, is lost in the limit. This is why we work with integral formulas instead of partial differential equations. The rigorous version of the above (standard) argument, which will be presented in Section 2.8, resorts to the convergence of (2.4.31) to (2.4.22), and requires only continuity of the limiting initial data $f_j(0)$.

2.5 Assumptions and results

We establish here the hypotheses under which we will work. Beyond Hypothesis 1 on the interaction potential stated in Section 2.3.2, we assume

Hypothesis 2. The initial condition for the Boltzmann equation is a continuous probability density f_0 over $\mathbb{R}^6$, satisfying the bound

$$\sup_{x,v} e^{\frac{\beta}{2}v^2} f_0(x,v) < +\infty \tag{2.5.1}$$

for some $\beta > 0$.

Moreover, indicating by $H(\mathbf{z}_j)$ the $j-$particle Hamiltonian written in macroscopic variables,

$$H(\mathbf{z}_j) = \frac{1}{2}\sum_{i=1}^{j} v_i^2 + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq k}}^{j} \Phi\left(\frac{x_i - x_k}{\varepsilon}\right) \,, \tag{2.5.2}$$

we have

Hypothesis 3. The initial probability density on $\mathcal{M}_N$ is a Borel function W_0^N symmetric in the particle labels. Its reduced marginals $f^N_{0,j}, j = 1, \cdots, N$ satisfy the bound

$$f^N_{0,j}(\mathbf{z}_j) e^{\beta H(\mathbf{z}_j)} \leq e^{\alpha j} \tag{2.5.3}$$

for some $\alpha > 0$.

By Proposition 1, the $f^N_{0,j}$ are good initial data for the evolutions (2.4.22) and (2.4.23). Note also that Hypothesis 3 implies that we are fixing correlations even at time zero. Indeed, if the interaction potential diverges at the origin, $f^N_{0,j}(\mathbf{z}_j) \to 0$ exponentially whenever $x_k \to x_i$ for $k \neq i$. Therefore, initial product states are excluded. This situation is similar to that of hard–sphere systems, in which an overlapping of any pair of particles is not allowed. Even if the potential is bounded, but positive at the origin (which is the case of stable interactions), product states are forbidden by Hypothesis 3. In fact, near the diagonal ($x_k = x_i$) the factor $e^{\beta H(\mathbf{z}_j)}$ can grow exponentially with j^2.

Our last hypothesis is

Hypothesis 4. The family $f_{0,j}^N$ converges to $f_{0,j} := f_0^{\otimes j}$ as $N \to \infty$, uniformly on compact sets in $\mathcal{M}_j$.

We are now ready to state our first result. Let $f_j^N(t)$ be the reduced marginals at time t, evolved according to Eq. (2.4.22) and let $f_j(t)$ be defined as in (2.4.27) and (2.4.31) (which will be proven to be an absolutely convergent series in Section 2.6). Define also the subset of particles that cannot collide pointwise under the free evolution:

$$\Omega_j = \left\{ \mathbf{z}_j \in \mathcal{M}_j \text{ s.t. } (x_i - x_k) \wedge (v_i - v_k) \neq 0 \right\} . \tag{2.5.4}$$

Theorem 1. *Under the Hypotheses 1–4, there exists* $t_0 > 0$ *such that, for* $0 < t < t_0$ *and* $j > 0$,

$$\lim_{\substack{\varepsilon \to 0 \\ N\varepsilon^2 = 1}} f_j^N(t) = f_j(t) \tag{2.5.5}$$

uniformly on compact sets in Ω_j.

Theorem 1 is formulated and proven in the same spirit of [10] and [9]. As we shall see in Section 2.8.2, the proof, based on geometrical arguments, is abstract and does not give informations on the rate of convergence. However, the result can be improved under quantitative assumptions on the rate of convergence and the continuity of the initial data, as explained in what follows.

Define

$$\mathcal{M}_j(\delta) = \left\{ \mathbf{z}_j \in \mathbb{R}^{6j} \;\middle|\; |x_i - x_k| > \delta,\ i, k = 1 \cdots j,\ k \neq i \right\} \tag{2.5.6}$$

for $\delta > 0$. We assume

Hypothesis 5. For some $C' > 0$,

$$\sup_{\mathbf{z}_j \in \mathcal{M}_j(\varepsilon)} e^{\frac{\beta}{2}\sum_{i=1}^j v_i^2} \left| f_{0,j}(\mathbf{z}_j) - f_{0,j}^N(\mathbf{z}_j) \right| \leq (C')^j \varepsilon \, . \tag{2.5.7}$$

Moreover, for some $L > 0$,

$$e^{\frac{\beta}{2}v^2} \left| f_0(x,v) - f_0(x',v) \right| \leq L \left| x - x' \right| . \tag{2.5.8}$$

Then we have the following:

Theorem 2. *Assume the Hypotheses* $1 - 5$*. Then, for all* $\mathbf{z}_j \in \Omega_j, j > 0$ *and* t *sufficiently small, there exists a positive* $\varepsilon_0 = \varepsilon_0(\mathbf{z}_j)$ *such that, for* $\varepsilon < \varepsilon_0$ *and* $N\varepsilon^2 = 1$

$$|f_j^N(\mathbf{z}_j, t) - f_j(\mathbf{z}_j, t)| \leq C^j \varepsilon^\gamma \, , \tag{2.5.9}$$

where $C > 0, \gamma > 0$ *are suitable constants.*

Observe that Hypotheses 4, 5 are a natural notion of convergence compatible with continuity of f_0 and estimate (2.5.3) (which prevents convergence on the diagonals $x_i = x_k$). To clarify this point, we construct some explicit example in the next subsection.

2.5.1 An example of initial datum

In the following we present a sequence of reduced marginals satisfying Hypotheses 3–5. Set

$$W_0^N(\mathbf{z}_N) = \frac{1}{\mathcal{Z}_N} f_0^{\otimes N}(\mathbf{z}_N) \prod_{1\le i<k\le N} \mathbb{1}_{\{|x_i-x_k|>\varepsilon\}}(\mathbf{z}_N)\ , \tag{2.5.10}$$

where

$$\mathcal{Z}_N = \int_{\mathbb{R}^{6N}} d\mathbf{z}_N f_0^{\otimes N}(\mathbf{z}_N) \prod_{1\le i<k\le N} \mathbb{1}_{\{|x_i-x_k|>\varepsilon\}}(\mathbf{z}_N) \tag{2.5.11}$$

is the partition function and f_0 is some density satisfying Hypothesis 2 and (2.5.8). The reduced marginals are

$$f_{0,j}^N(\mathbf{z}_j) = \frac{F^N(\mathbf{z}_j)}{\mathcal{Z}_N} f_0^{\otimes j}(\mathbf{z}_j) \prod_{1\le i<k\le j} \mathbb{1}_{\{|x_i-x_k|>\varepsilon\}}(\mathbf{z}_j) \tag{2.5.12}$$

with

$$\begin{aligned} F^N(\mathbf{z}_j) = \int_{\mathbb{R}^{6(N-j)}} d\mathbf{z}_{j,N} f_0^{\otimes(N-j)}(\mathbf{z}_{j,N}) &\left(\prod_{i=1}^{j}\prod_{k=j+1}^{N} \mathbb{1}_{\{|x_i-x_k|>\varepsilon\}}(\mathbf{z}_N)\right) \\ &\cdot\left(\prod_{j+1\le i<k\le N} \mathbb{1}_{\{|x_i-x_k|>\varepsilon\}}(\mathbf{z}_{j,N})\right)\ . \end{aligned} \tag{2.5.13}$$

Let us estimate $F^N(\mathbf{z}_j)\mathcal{Z}_N^{-1}$. First observe that, for some $C_0>0$,

$$\mathcal{Z}_{N-j}(1-C_0N\varepsilon^3)^j \le \mathcal{Z}_N \le \mathcal{Z}_{N-j}\ . \tag{2.5.14}$$

The upper bound is obvious consequence of the normalization of f_0. As regards the lower bound, note that

$$\begin{aligned} \mathcal{Z}_N &= \int_{\mathbb{R}^{6(N-1)}} d\mathbf{z}_{N-1} f_0^{\otimes(N-1)}(\mathbf{z}_{N-1}) \prod_{1\le i<k\le N-1} \mathbb{1}_{\{|x_i-x_k|>\varepsilon\}}(\mathbf{z}_{N-1}) \\ &\quad\cdot \int_{\mathbb{R}^6} dz_N f_0(z_N) \prod_{i=1}^{N-1} \mathbb{1}_{\{|x_i-x_N|>\varepsilon\}}(z_N) \\ &= \int_{\mathbb{R}^{6(N-1)}} d\mathbf{z}_{N-1} f_0^{\otimes(N-1)}(\mathbf{z}_{N-1}) \prod_{1\le i<k\le N-1} \mathbb{1}_{\{|x_i-x_k|>\varepsilon\}}(\mathbf{z}_{N-1}) \\ &\quad\cdot \int_{\mathbb{R}^6} dz_N f_0(z_N) \left(1-\sum_{i=1}^{N-1} \mathbb{1}_{\{|x_i-x_N|\le\varepsilon\}}(z_N)\right) \\ &\ge \mathcal{Z}_{N-1}(1-C_0(N-1)\varepsilon^3)\ , \end{aligned} \tag{2.5.15}$$

for instance taking $C_0 = (4\pi/3)\|f_0\|_\infty$. Eq. (2.5.14) follows by iteration. We can also show that

$$\mathcal{Z}_{N-j}\left(1-j\frac{C_0N\varepsilon^3}{1-C_0N\varepsilon^3}\right) \le F^N(\mathbf{z}_j) \le \mathcal{Z}_{N-j}\ . \tag{2.5.16}$$

The upper bound is immediate, while the lower bound follows from

$$\begin{aligned} F^N(\mathbf{z}_j) &\geq \int_{\mathbb{R}^{6(N-j)}} d\mathbf{z}_{j,N} f_0^{\otimes(N-j)}(\mathbf{z}_{j,N}) \left(1 - \sum_{i=1}^{j}\sum_{k=j+1}^{N} \mathbb{1}_{\{|x_i-x_k|\leq\varepsilon\}}(\mathbf{z}_N)\right) \\ &\qquad\qquad \cdot \left(\prod_{j+1\leq i<k\leq N} \mathbb{1}_{\{|x_i-x_k|>\varepsilon\}}(\mathbf{z}_{j,N})\right) \\ &\geq \mathcal{Z}_{N-j} - j(N-j)C_0\varepsilon^3 \mathcal{Z}_{N-j-1} \ , \\ &\geq \mathcal{Z}_{N-j}\left(1 - jNC_0\varepsilon^3 \frac{\mathcal{Z}_{N-j-1}}{\mathcal{Z}_{N-j}}\right) \ , \end{aligned} \tag{2.5.17}$$

noticing that (2.5.14) implies $\mathcal{Z}_{N-j-1}\mathcal{Z}_{N-j}^{-1} \leq (1-C_0N\varepsilon^3)^{-1}$. If $N\varepsilon^2 = 1$ and N is sufficiently large, Equations (2.5.14) and (2.5.16) give in turn the bounds

$$1 - 2C_0 j\varepsilon \leq \frac{F^N(\mathbf{z}_j)}{\mathcal{Z}_N} \leq \frac{1}{(1-C_0\varepsilon)^j} \tag{2.5.18}$$

and, in particular,

$$\frac{F^N(\mathbf{z}_j)}{\mathcal{Z}_N} \underset{N\to\infty}{\longrightarrow} 1 \tag{2.5.19}$$

uniformly in $\mathbf{z}_j \in \mathcal{M}_j$.

Now it is easy to check that, for N sufficiently large, the Hypotheses 3–5 are verified. Hypothesis 3 follows from Hypothesis 2. Hypothesis 5 (hence 4) follows from the estimates in (2.5.18).

In definition (2.5.10) for the initial density we could also replace the product of characteristic functions by $e^{-\beta\sum_{i<k}\Phi\left(\frac{x_i-x_k}{\varepsilon}\right)}$, see [5]. This defines a sequence of states which are, in a sense, the maximally uncorrelated states for which the Hypotheses are satisfied.

Finally, other families of initial data exhibiting a slower rate of convergence (and implying possibly a slower convergence in Theorem 2) can be easily constructed, for instance enlarging the cut–off in (2.5.10). If in formula (2.5.10) ε is replaced by $\varepsilon^{\gamma'}$ with $\gamma' \in (2/3, 1]$, then, proceeding as before, we obtain

$$\sup_{\mathbf{z}_j\in\mathcal{M}_j(\varepsilon^{\gamma'})} \left|f_{0,j}(\mathbf{z}_j) - f_{0,j}^N(\mathbf{z}_j)\right| \leq (C')^j \varepsilon^{-2+3\gamma'} \ . \tag{2.5.20}$$

2.5.2 General strategy of the proof

The proof follows the main ideas of [10], adapted to the present context. The validity argument is based on a comparison among the series for the $N-$particle system (2.4.22), and the Boltzmann series (2.4.31).

- First, we prove that both the expansions are absolutely convergent series, for sufficiently short times and uniformly in ε : see Section 2.6. As a consequence of the estimates in Section 2.6, it follows also that (2.4.22) and (2.4.23) are asymptotically equivalent in the BG limit.

- Then, it remains to prove the term by term convergence of (2.4.23) to (2.4.31). To do this, it is necessary a preliminary detailed analysis of the generic term. This is presented in Sections 2.7 and 2.7.1 for the series (2.4.23), and in Section 2.7.2 for the Boltzmann series (2.4.31). The structure of the generic term is described with the help of a convenient representation of formulas in terms of tree graphs. It turns out that any given term can be expressed as an integral over a set of special backwards–in–time trajectories of clusters of particles.

- The proof of the term by term convergence is carried out in Section 2.8, using in a crucial way the picture introduced in Section 2.7. The issues arising from the convergence will be first discussed in Section 2.8.1, while the abstract proof leading to Theorem 1 and the explicit estimates leading to Theorem 2 will be presented in Sections 2.8.2 and 2.8.3 respectively.

2.6 Short time estimates

The aim of this section is to prove that, for times t smaller than a certain t_0, the expansion for $f_j^N(t)$, Eq. (2.4.22), can be bounded uniformly in ε. The Boltzmann series solution Eq. (2.4.31) turns out to be an absolutely convergent series for the same values of t. Moreover, the difference between $f_j^N(t)$ and $\tilde{f}_j^N(t)$ (defined by (2.4.23)) is negligible in the limit. These results are easily established by assuming the bounds on the initial data in Hypothesis 2 and 3. Here we will follow [9] (see also [5, 19, 18, 3]).

To begin with, we notice that our assumptions on the initial data make natural the introduction of the norms:

$$\begin{aligned} \|g_j\|_\beta &= \sup_{\mathbf{z}_j} e^{\beta H(\mathbf{z}_j)} |g_j(\mathbf{z}_j)| \ , \qquad && g_j : \mathcal{M}_j \to \mathbb{R} \ , \quad \beta > 0 \ , \\ \|g\|_{\beta,\alpha} &= \sup_{j\geq 1} e^{-\alpha j} \|g_j\|_\beta \ , && g = \{g_j\}_{j=1}^\infty \ , \qquad \alpha > 0 \ . \end{aligned} \tag{2.6.1}$$

By the energy conservation

$$\|\mathcal{S}^\varepsilon(t) g\|_{\beta,\alpha} = \|g\|_{\beta,\alpha} \ , \tag{2.6.2}$$

for all β and α for which the right hand side makes sense.

The crucial estimate is the following:

Lemma 3. *Let $g_j^N : \mathcal{M}_j \to \mathbb{R}$ be a sequence of continuous[1] functions with $g_j^N = 0$ for $j > N$ and satisfying the estimate of Hypothesis 3. Set $\mathcal{A}^\varepsilon g^N = \left\{ \sum_{m\geq 0} \mathcal{A}^\varepsilon_{j+1+m} g^N_{j+1+m} \right\}_{j=1}^\infty$. Then, for $\beta' < \beta$ and $\alpha' > \alpha$ there exists a pure constant $\bar{C} > 0$ such that, for ε small enough,*

$$\|\mathcal{A}^\varepsilon g^N\|_{\beta',\alpha'} \leq \bar{C} \left(\frac{1}{\sqrt{(\beta-\beta')(\alpha'-\alpha)}} + \frac{1}{\alpha'-\alpha} \right) \|g^N\|_{\beta,\alpha} \ . \tag{2.6.3}$$

[1]The continuity here is required only for simplicity of notation, since it assures well posedness of the operator action: see the Remark after Proposition 1. If that is not true, the lemma must be reformulated for $\int_0^t ds \frac{s^{n-1}}{(n-1)!} \mathcal{A}^\varepsilon \mathcal{S}^\varepsilon(s)$ (that is what we really need to control for the proof of Proposition 2 below). This can be done in an obvious way using Eq. (2.6.2) and adding a factor $t^n/n!$ in the right hand side of the estimate.

Proof. From definition (2.4.17) we find

$$
\begin{aligned}
e^{\beta' H(\mathbf{z}_j)} |\mathcal{A}^{\varepsilon}_{j+1+m} g^N_{j+1+m}(\mathbf{z}_j)| \leq (N-j-1)\cdots(N-j-m) \sum_{i=1}^{j} \int_{S^2} d\nu \|g^N_{j+1+m}\|_\beta \\
\cdot \int dv_{j+1}(|v_i| + |v_{j+1}|) e^{-(\beta-\beta')H(\mathbf{z}_j)} e^{-\frac{\beta}{2}v^2_{j+1}} \\
\cdot \int_{\Delta_m(\mathbf{x}_{j+1})} \frac{d\mathbf{z}_{j+1,m}}{m!} e^{-\frac{\beta}{2}\sum_{i=j+2}^{j+1+m} v_i^2} .
\end{aligned}
\tag{2.6.4}
$$

Here we used that $\varepsilon^2(N-j) \leq 1$ and the positivity of the interaction (Hypothesis 1), for which

$$
H(\mathbf{z}_{j+1+m}) = H(\mathbf{z}_j) + H(\mathbf{z}_{j,1+m}) \geq H(\mathbf{z}_j) + \frac{1}{2}\sum_{i=j+1}^{j+1+m} v_i^2 . \tag{2.6.5}
$$

The last integral in the right hand side is bounded by

$$
\left(\frac{2\pi}{\beta}\right)^{\frac{3}{2}m} \left(\frac{4\pi}{3}\right)^m \varepsilon^{3m} , \tag{2.6.6}
$$

so that (2.6.4) is smaller than

$$
\|g^N_{j+1+m}\|_\beta (C_\beta \varepsilon)^m \sum_{i=1}^{j} \int dv_{j+1}(|v_i| + |v_{j+1}|) e^{-\frac{\beta-\beta'}{2}\sum_{i=1}^{j} v_i^2} e^{-\frac{\beta}{2}v^2_{j+1}} , \tag{2.6.7}
$$

where we used again positivity of the interaction, and C_β is a suitable constant. The Cauchy–Schwarz inequality implies

$$
\sum_{i=1}^{j}(|v_i| + |v_{j+1}|) \leq \sqrt{j\sum_{i=1}^{j} v_i^2} + j|v_{j+1}|, \tag{2.6.8}
$$

which inserted into (2.6.7) leads to

$$
\|\mathcal{A}^{\varepsilon}_{j+1+m} g^N_{j+1+m}\|_{\beta'} \leq (C'_\beta \varepsilon)^m \|g^N_{j+1+m}\|_\beta \left(\frac{\sqrt{j}}{\sqrt{\beta-\beta'}} + j \right) . \tag{2.6.9}
$$

Summing over m and taking the supremum over j with weight $e^{-\alpha' j}$ we readily get (2.6.3) for ε smaller than a constant depending only on β, α. □

Let us apply Lemma 3, together with (2.6.2), to the right hand side of (2.4.22). We proceed by iteration. For a given $n > 0$, we partition the intervals $[\beta/2, \beta]$ and $[\alpha, 2\alpha]$ in n intervals of the same length $\frac{\beta}{2n}$ and $\frac{\alpha}{n}$, and then apply the above results n times. The outcome is

$$
\begin{aligned}
\|f^N(t)\|_{\beta/2,2\alpha} &\leq \sum_{n\geq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n (C_{\beta,\alpha} n)^n \|f^N_0\|_{\beta,\alpha} \\
&= \sum_{n\geq 0} \frac{t^n}{n!} (C_{\beta,\alpha} n)^n \|f^N_0\|_{\beta,\alpha} \\
&\leq \|f^N_0\|_{\beta,\alpha} \sum_{n\geq 0} (t C'_{\beta,\alpha})^n ,
\end{aligned}
\tag{2.6.10}
$$

for suitable constants $C_{\beta,\alpha}, C'_{\beta,\alpha}$, having used Stirling formula in the last step. Hence we obtained a geometric series which converges for t sufficiently small (and the radius of convergence is explicitly computable in terms of the other constants). Now the same argument can be applied in a straightforward way to the series (2.4.23) and (2.4.31). Thus, we have proven the first statement of:

Proposition 2. *In the Hypotheses 2 and 3, we have absolute convergence of the series* (2.4.22), (2.4.23) *(uniformly in the BG limit for ε small enough) and* (2.4.31), *for all $t < t_0 = t_0(\beta, \alpha)$. Moreover, for some $C'' > 0$, if ε is small enough,*

$$\|f^N(t) - \tilde{f}^N(t)\|_{\beta/2,2\alpha} \leq C''\varepsilon \, . \tag{2.6.11}$$

Proof. We just need to prove Eq. (2.6.11). Set $\mathcal{C}^\varepsilon g^N = \left\{ \varepsilon^2 (N-j) \mathcal{C}^\varepsilon_{j+1} g^N_{j+1} \right\}_{j=1}^\infty$. With the notations of Lemma 3 and proceeding in the same way, we observe that

$$\begin{aligned} \|(\mathcal{A}^\varepsilon - \mathcal{C}^\varepsilon) g^N\|_{\beta',\alpha'} &\leq \sup_{j\geq 1} e^{-\alpha' j} \sum_{m \geq 1} \|\mathcal{A}^\varepsilon_{j+1+m} g^N_{j+1+m}\|_{\beta'} \\ &\leq \sup_{j\geq 1} e^{-\alpha' j} \sum_{m\geq 1} (C'_\beta \varepsilon)^m \|g^N_{j+1+m}\|_\beta \left(\frac{\sqrt{j}}{\sqrt{\beta - \beta'}} + j \right) \\ &\leq C''_{\beta,\alpha} \varepsilon \left(\frac{1}{\sqrt{(\beta-\beta')(\alpha'-\alpha)}} + \frac{1}{\alpha' - \alpha} \right) \|g^N\|_{\beta,\alpha} \end{aligned} \tag{2.6.12}$$

for suitable $C''_{\beta,\alpha} > 0$, having used (2.6.9) in the first inequality, and ε sufficiently small in the second. Therefore, proceeding as in (2.6.10),

$$\begin{aligned} \|f^N(t) - \tilde{f}^N(t)\|_{\beta/2,2\alpha} &\leq \sum_{n\geq 0} \frac{t^n}{n!} \sum_{k=1}^n \binom{n}{k} \varepsilon^k (C_{\beta,\alpha} n)^n \|f^N_0\|_{\beta,\alpha} \\ &\leq \varepsilon \|f^N_0\|_{\beta,\alpha} \sum_{n\geq 0} (t 2 C'_{\beta,\alpha})^n \, , \end{aligned} \tag{2.6.13}$$

which gives the result with C'' depending on β, α and on the initial datum. □

2.7 The tree expansion

In the proof of Theorems 1 and 2 it is convenient to represent each term of the expansions (2.4.23) and (2.4.31) as more explicit integrals of the initial data, $f^N_{0,j}$ and $f_{0,j}$ respectively. As we will see in the present section, it is natural to express such terms by means of binary trees which help us to visualize the various contributions.

Consider first Eq. (2.4.23) which, reminding Eq. (2.4.25), we rewrite as

$$\begin{aligned} \tilde{f}^N_j(t) = \sum_{n=0}^{N-j} \alpha^\varepsilon_n(j) \sum_{\boldsymbol{\sigma}_n} \sum_{\mathbf{k}_n}{}^* (-1)^{|\boldsymbol{\sigma}_n|} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ \cdot \mathcal{S}^\varepsilon_j(t-t_1) \mathcal{C}^{\varepsilon,\sigma_1}_{k_1,j+1} \mathcal{S}^\varepsilon_{j+1}(t_1-t_2) \cdots \mathcal{C}^{\varepsilon,\sigma_n}_{k_n,j+n} \mathcal{S}^\varepsilon_{j+n}(t_n) f^N_{0,j+n} \, , \end{aligned} \tag{2.7.1}$$

where

$$\boldsymbol{\sigma}_n = (\sigma_1, \cdots, \sigma_n), \qquad \sigma_i = \pm \ , \qquad |\sigma_n| = \sum_{i=1}^{n} \sigma_i 1 \ ,$$

$$\sum_{\mathbf{k}_n}^{*} = \sum_{k_1=1}^{j} \sum_{k_2=1}^{j+1} \cdots \sum_{k_n=1}^{j+n-1} \ . \tag{2.7.2}$$

We introduce the *$n-$collision, $j-$particle tree graph*, denoted $\Gamma(j,n)$, as the collection of integers $k_1, \cdots, k_n$ that are present in the sum (2.7.2), i.e.

$$k_1 \in I_j, k_2 \in I_{j+1}, \cdots, k_n \in I_{j+n} \ , \qquad \text{with} \qquad I_s = \{1, 2, \cdots, s\}, \tag{2.7.3}$$

so that we shall write

$$\sum_{\mathbf{k}_n}^{*} = \sum_{\Gamma(j,n)} \ . \tag{2.7.4}$$

Note that the number of terms in the sum is $j(j+1)\cdots(j+n-1)$. The name *tree graph* is justified by the fact that it has a natural graphical representation. This is best explained by an example: see Figure 2.2 which corresponds to $\Gamma(2,5)$ given by $1,2,1,3,2$.

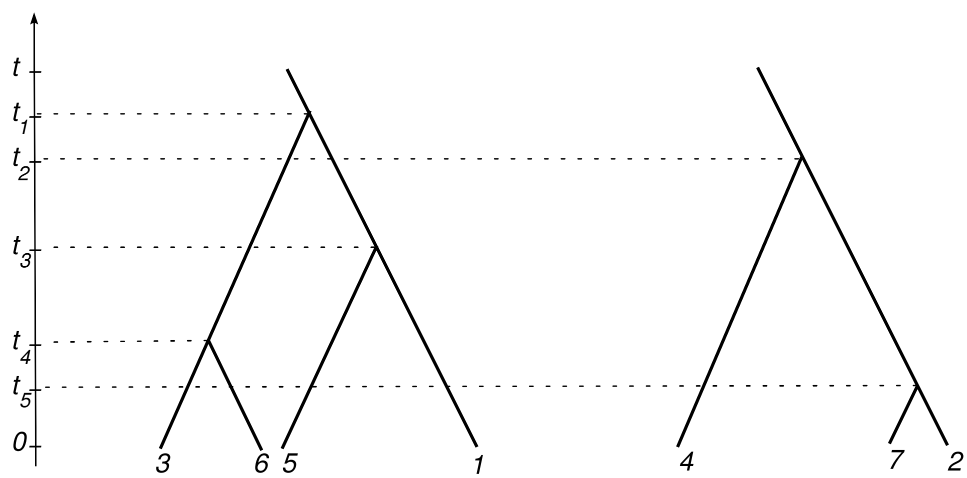

Figure 2.2: Tree graph $\Gamma(2,5) = 1,2,1,3,2$. We have also drawn a time arrow in order to associate times to the nodes of the trees: at time t_i the line $j+i$ is"created". Lines 1 and 2 exist for all times; they are called"root lines".

Given a tree graph $\Gamma(j,n)$, and fixed a value of $\boldsymbol{\sigma}_n$ and of all the integration variables in the expansion (2.7.2) (times, unit vectors, velocities), we can associate to it a special ($\varepsilon-$dependent) trajectory of particles, which we call *interacting backwards flow* (IBF in the following), since it will be naturally defined by going back in time. The rules for the construction of this evolution will be explained in the next section. The notation for a configuration of particles in the IBF will make use of Greek alphabet i.e. $\boldsymbol{\zeta}^{\varepsilon}(s)$, where $s \in [0,t]$ is the time and there is *no* label specifying the number of particles. If $s \in (t_{r+1}, t_r)$ (with the convention $t_0 = t, t_{n+1} = 0$) we have $j + r$ particles:

$$\boldsymbol{\zeta}^\varepsilon(s) = (\zeta_1^\varepsilon(s), \cdots, \zeta_{j+r}^\varepsilon(s)) \in \mathcal{M}_{j+r} \qquad \text{for } s \in (t_{r+1}, t_r) \ , \tag{2.7.5}$$

with

$$\zeta_i^\varepsilon(s) = (\xi_i^\varepsilon(s), \eta_i^\varepsilon(s)) \ , \tag{2.7.6}$$

the positions and velocities of all the particles being respectively

$$\begin{aligned} \boldsymbol{\xi}^\varepsilon(s) &= (\xi_1^\varepsilon(s), \cdots, \xi_{j+r}^\varepsilon(s)) \ , \\ \boldsymbol{\eta}^\varepsilon(s) &= (\eta_1^\varepsilon(s), \ldots, \eta_{j+r}^\varepsilon(s)) \ . \end{aligned} \tag{2.7.7}$$

The reason to introduce these trajectories is that we want a more explicit expression of each term of the expansion (2.7.1), namely our purpose is to write Eq. (2.7.1) as

$$\tilde{f}_j^N(\mathbf{z}_j, t) = \sum_{n=0}^{N-j} \alpha_n^\varepsilon(j) \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} (-1)^{|\boldsymbol{\sigma}_n|} \mathcal{T}^\varepsilon_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t) \ , \tag{2.7.8}$$

where

$$\mathcal{T}^\varepsilon_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t) = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \prod_{i=1}^n B^\varepsilon(\nu_i; v_{j+i} - \eta^\varepsilon_{k_i}(t_i)) f^N_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)) \ , \tag{2.7.9}$$

$d\Lambda$ is the measure on $\mathbb{R}^n \times S^{2n} \times \mathbb{R}^{3n}$ given by

$$d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) = \mathbb{1}_{\{t_1 > t_2 \cdots > t_n\}} dt_1 \ldots dt_n d\nu_1 \ldots d\nu_n dv_{j+1} \ldots dv_{j+n} \ , \tag{2.7.10}$$

and we use the short notation

$$B^\varepsilon(\nu_i; v_{j+i} - \eta^\varepsilon_{k_i}(t_i)) = |\nu_i \cdot (v_{j+i} - \eta^\varepsilon_{k_i}(t_i))| \mathbb{1}_{\{\sigma_i \nu_i \cdot (v_{j+i} - \eta^\varepsilon_{k_i}(t_i)) \geq 0\}} \mathbb{1}_{\{|\xi^\varepsilon_{j+i}(t_i) - \xi^\varepsilon_k(t_i)| > \varepsilon \ \forall k \neq k_i\}} \ . \tag{2.7.11}$$

In other words, in the generic term $\mathcal{T}^\varepsilon_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t)$, the initial datum $f^N_{0,j+n}$ is integrated, with the suitable weight, over all the possible time–zero states of the IBF associated to $\Gamma(j,n), \boldsymbol{\sigma}_n$.

2.7.1 The interacting backwards flow (IBF)

Let us construct $\boldsymbol{\zeta}^\varepsilon(s)$ for a fixed collection of variables $\Gamma(j,n), \boldsymbol{\sigma}_n, \mathbf{z}_j, \mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}$, with

$$t \equiv t_0 > t_1 > t_2 > \cdots > t_n > t_{n+1} \equiv 0 \ , \tag{2.7.12}$$

and $\boldsymbol{\nu}_n$ satisfying a further constraint that will be specified below. The j root lines of the tree graph are associated to the first j particles, with states $\zeta_1^\varepsilon, \cdots, \zeta_j^\varepsilon$. Each branch $j + \ell$ ($\ell = 1, \cdots, n$) represents a new particle with the same label, and state $\zeta^\varepsilon_{j+\ell}$. This new particle appears, going backwards in time, at time t_ℓ in a collision state with a previous particle (branch) $k_\ell \in \{1, \cdots j+\ell-1\}$, with either incoming or outgoing velocity according to $\sigma_\ell = -$ or $\sigma_\ell = +$ respectively.

More precisely, in the time interval (t_r, t_{r-1}) particles $1, \cdots, j+r-1$ flow according to the usual dynamics $\mathtt{T}^\varepsilon_{j+r-1}$. This defines $\boldsymbol{\zeta}^\varepsilon_{j+r-1}(s)$ starting from $\boldsymbol{\zeta}^\varepsilon_{j+r-1}(t_{r-1})$. At time t_r the particle $j+r$ is "*created*" by particle k_r in the position

$$\xi^\varepsilon_{j+r}(t_r) = \xi^\varepsilon_{k_r}(t_r) + \nu_r \varepsilon \tag{2.7.13}$$

and with velocity v_{j+r}. This defines $\boldsymbol{\zeta}^\varepsilon(t_r) = (\zeta^\varepsilon_1(t_r), \cdots, \zeta^\varepsilon_{j+r}(t_r))$. After that, the evolution in (t_{r+1}, t_r) is contructed applying to this configuration the dynamics $\mathtt{T}^\varepsilon_{j+r}$ (with negative times). The characteristic function in the collision operator (2.4.25) (or the second characteristic function in (2.7.11)), is a constraint on ν_r implying that no third particle is closer than ε to the pair $k_r, j+r$ at the time t_r.

We have two cases. If $\sigma_r = -$, then it must be $\nu_r \cdot (v_{j+r} - \eta^\varepsilon_{k_r}(t_r)) \leq 0$. In this case the velocities are incoming and no scattering occurs, namely after t_r the pair of particles moves backwards freely with velocities $\eta(t_r)$ and v_{j+r}. If $\sigma_r = +$, we require $\nu_r \cdot (v_{j+r} - \eta^\varepsilon_{k_r}(t_r)) \geq 0$ so that the pair is post–collisional. Then the presence of the interaction in the flow $\mathtt{T}^\varepsilon_{j+r}$ forces the pair to perform a (backwards) scattering. The two situations are illustrated in Fig. 2.3.

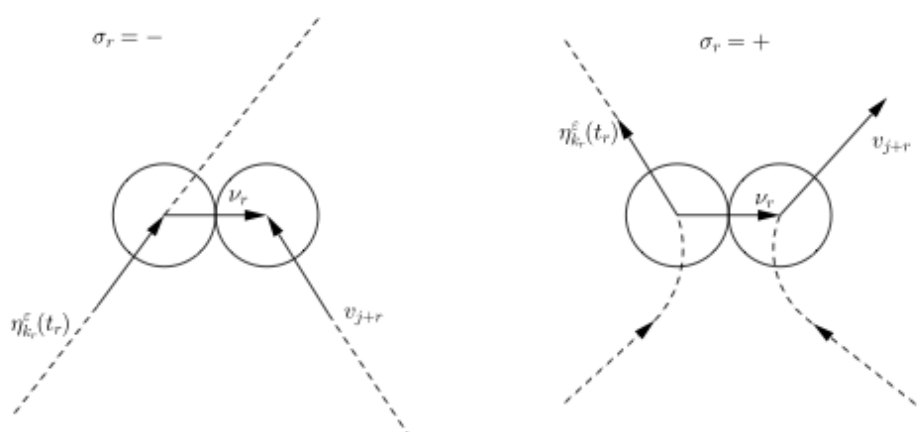

Figure 2.3: At time t_r, particle $j+r$ is *created* by particle k_r, either in incoming ($\sigma_r = -$) or in outgoing ($\sigma_r = +$) collision configuration. Particle k_r is called *progenitor* of particle $j+r$.

Remark. It is very important to note that between two creation times t_r, t_{r+1} any pair of particles among the $j+r$, different from the couple $(k_r, j+r)$, can possibly interact by reaching (or having from the beginning) a distance smaller than ε. These interactions are called *recollisions*, because they generally involve particles that have already interacted at some creation time (in the future) with another particle of the IBF. In our language, recollisions are the "interactions different from creations". Though recollisions are expected to be unlikely, we will have to analyze them with special care, since they are the main responsible of the different behavior of the particle dynamics from the Boltzmann evolution.

2.7.2 The Boltzmann backwards flow (BBF)

The discussion of the two previous sections can be repeated, with minor changes, for the case of Boltzmann series (2.4.31). The interacting backwards flow is now substituted by the *Boltzmann backwards flow* (BBF) $\boldsymbol{\zeta}(s)$, for which we use the same notations of (2.7.5)–(2.7.7) with the superscript ε omitted. The BBF is introduced exactly as the IBF, see Section 2.7.1, except for the following differences:

- the interacting dynamics $\mathtt{T}^\varepsilon$ is replaced by the simple free dynamics;
- in the right hand side of (2.7.13) the second term is missing, i.e. the created particle appears at the same position of its progenitor;
- there is no constraint on ν_r other than the one implied by the value of σ_r;
- if $\sigma_r = +$, to determine the state of particles in (t_{r+1}, t_r), *before* applying free evolution we have to change velocities according to $(\eta_{k_r}(t_r^+), v_{j+r}) \to (\eta_{k_r}(t_r^-), \eta_{j+r}(t_r^-))$, where $\to$ denotes the scattering rule depicted in (2.2.2) and Figure 2.1. Here $\eta_{k_r}(t_r^+)$ indicates the limit from the future, and $\eta_{k_r}(t_r^-)$ the limit from the past.

Eq. (2.4.31) can then be rewritten:

$$f_j(\mathbf{z}_j, t) = \sum_{n=0}^{\infty} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} (-1)^{|\boldsymbol{\sigma}_n|} \mathcal{T}_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t) \;, \tag{2.7.14}$$

where

$$\mathcal{T}_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t) = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \prod_{i=1}^{n} B(\nu_i; v_{j+i} - \eta_{k_i}(t_i^+)) f_{0,j+n}(\boldsymbol{\zeta}(0)) \;, \tag{2.7.15}$$

and

$$B(\nu_i; v_{j+i} - \eta_{k_i}(t_i^+)) = |\nu_i \cdot (v_{j+i} - \eta_{k_i}(t_i^+))| \mathbb{1}_{\{\sigma_i \nu_i \cdot (v_{j+i} - \eta_{k_i}(t_i^+)) \geq 0\}} \;. \tag{2.7.16}$$

Remark. An important point concerning the Boltzmann backwards flow defined in the previous section is that, in a given interval (t_{r+1}, t_r), the velocities $\eta_i(s) = \eta_i(t_r^-)$, besides being constant, depend only on the velocities of particles in the future of the BBF, and on the vectors of impact $\nu_1, \cdots, \nu_r$, but *not* on the interaction times $t_1, \cdots, t_r$. This simple structure of the BBF will be exploited later on (see for instance Equation (2.8.13)).

In the proof of the term by term convergence it will be used a change of variables transforming integrals over outgoing variables into integrals over incoming variables. This is simply the scattering operator of (2.3.24) applied to an interaction of the BBF. We conclude this subsection introducing such operation.

Fix $\Gamma(j,n)$, $1 \le r \le n$, $\mathbf{v}_j \in \mathbb{R}^{3j}$ and define the transformation $\mathcal{I}^{(r)} = \mathcal{I}^{(r)}_{\mathbf{v}_j,\Gamma(j,n)}$:

$$\begin{aligned}
&\mathcal{I}^{(r)} : S^{2n} \times \mathbb{R}^{3n} \longrightarrow S^{2n} \times \mathbb{R}^{3n} \\
&\mathcal{I}^{(r)}(\boldsymbol{\nu}_n, \mathbf{v}_{j,n}) = (\boldsymbol{\nu}_{r-1}, \nu'_r, \boldsymbol{\nu}_{r,n-r}, \mathbf{v}_{j,r-1}, V'_r, \mathbf{v}_{j+r,n-r})
\end{aligned} \tag{2.7.17}$$

where only the $r-$th couple (ν_r, v_{j+r}) is changed according to

$$\begin{cases}
\nu'_r = -\nu_r + 2\omega_r(\omega_r \cdot \nu_r) & \text{for } \nu_r \cdot (v_{j+r} - \eta_{k_r}(t_r^+)) > 0 \\
\nu'_r = \nu_r & \text{for } \nu_r \cdot (v_{j+r} - \eta_{k_r}(t_r^+)) \le 0 \\
V'_r = \eta_{j+r}(t_r^-) - \eta_{k_r}(t_r^-)
\end{cases} . \tag{2.7.18}$$

Here $\omega_r = \omega(\nu_r, v_{j+r} - \eta_{k_r}(t_r^+))$.

Lemma 4. *The transformation $\mathcal{I}^{(r)}$ is a one–to–one, measure preserving map.*

Proof. $\mathcal{I}^{(r)}$ is the composition of the two transformations:

$$\begin{aligned}
&(\nu_r, v_{j+r}) \longrightarrow (\nu_r, V_r) \\
&V_r = v_{j+r} - \eta_{k_r}(t_r^+)
\end{aligned} \tag{2.7.19}$$

and

$$(\nu_r, V_r) \longrightarrow (\nu'_r, V'_r) = \mathcal{I}^{-1}(\nu_r, V_r) \; , \tag{2.7.20}$$

where $\mathcal{I}^{-1}$ is the inverse scattering operator defined in Section 2.3.2 (in the case $\nu_r \cdot V_r \le 0$, just replace $\mathcal{I}^{-1}$ with the identity). The first is a simple translation by the vector $\eta_{k_r}(t_r^+) = \eta_{k_r}(t_{r-1}^-)$, which is a function of $\boldsymbol{\nu}_{r-1}, \mathbf{v}_{j,r-1}$ (see the Remark above). Therefore the result follows applying Lemma 2. □

2.8 Proof of the results

According to the strategy of Lanford, once proven the uniform convergence of the two series (2.4.22) and (2.4.31) for short times, we shall conclude the validity results, namely the convergence of $f_j^N(t)$ to $f_j(t)$, just proving the term by term convergence. Actually, by virtue of Proposition 2 in Section 2.6, it is enough to prove the term by term convergence of the series (2.4.23) to (2.4.31).

In Section 2.7 we have rephrased such expansions respectively in (2.7.8) and (2.7.14), i.e. sums over binary tree graphs of integrals over the (interacting or Boltzmann) backwards flows associated to the graph. Hence we must show convergence of the generic integral of this kind, $\mathcal{T}^{\varepsilon}_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t)$, to its analogue in the Boltzmann series, $\mathcal{T}_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t)$. The present section is devoted to this problem.

We stress once again the importance of the formulation of Grad (introduced in Section 2.4) which has been our starting point. In the language of Section 2.7 we could say that the terms in (2.4.22) that are absent in (2.4.23) collect all the interacting backwards flows in which two or more particles are created at some time t_i (graphically, three or more lines emerge from a node of the tree). The use of reduced marginals has allowed to identify all these negligible terms and to isolate them from the contributions of order one, namely $\alpha_n^\varepsilon(j)\mathcal{T}^\varepsilon_{\boldsymbol{\sigma}_n}(\mathbf{z}_j,t)$. Now looking at (2.7.9), we see that these last object resembles very much the generic term in the series solution of the BBGKY hierarchy for hard spheres. Nevertheless, as we will explain in the next subsection, in the case of smooth interactions one has to be more careful in studying the behavior of $\mathcal{T}^\varepsilon_{\boldsymbol{\sigma}_n}(\mathbf{z}_j,t)$ for ε small.

2.8.1 The convergence problem: preliminary considerations

Let us focus on $\mathcal{T}^\varepsilon_{\boldsymbol{\sigma}_n}(\mathbf{z}_j,t)$ and $\mathcal{T}_{\boldsymbol{\sigma}_n}(\mathbf{z}_j,t)$. The integrand functions depend on the variables $\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}$ completely through the trajectories of the IBF and the BBF respectively. In particular, the initial data $f^N_{0,j+n}$ and $f_{0,j+n}$ are integrated over the time–zero configurations of the flows. Since $f^N_{0,j+n}$ converges to $f_{0,j+n}$ by hypothesis, we must focus on the trajectories and prove that the IBF converges to the BBF for all values of $\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}$ outside a set giving a negligible contribution to the integrals.

Looking carefully at the definition of $\boldsymbol{\zeta}^\varepsilon(s)$ and $\boldsymbol{\zeta}(s)$ (see Sections 2.7.1 and 2.7.2), we realize that a great difference between them is generally caused by one of the following events:

1. a particle (say $j+i$) created in the IBF interacts for a very long time (i.e. larger than $O(\varepsilon)$) with its progenitor;

2. a couple of particles (i,h) of the IBF undergoes a recollision, i.e. an interaction different from a creation (see the remark on page 48);

3. a particle has a very large velocity, so that small differences between the two flows become large in a time of order 1.

Item 1, which is obviously absent in the case of hard spheres, is controlled by cutting off the variables (ν_i, v_{j+i}) that lead to the singular scattering, and showing that they give a small contribution to the integrals. Here the main technical issue is an estimate of the time of interaction, such as that of Lemma 1 (or its generalizations in the cases of potentials with an attractive part discussed in Section 2.9). Similarly, item 3 is controlled by cutting off the energy of the system, i.e. the large values of $|\mathbf{v}_{j,n}|$. Item 2 is the most delicate. It requires to demonstrate that the contribution of recolliding trajectories is negligible in the limit $\varepsilon \to 0$.

To motivate our strategy in controlling the recollisions, we start by the heuristic analysis of one of the simplest non–trivial cases, namely that in Figure 2.4. At time t particles 1 and 2 are in the final configuration $\mathbf{z}_2 = (x_1, v_1, x_2, v_2) \in \mathcal{M}_2$. We assume that the IBF is free up to time t_1, when particle 3 appears with velocity v_3 at distance $\varepsilon\nu_1$ from particle 1,

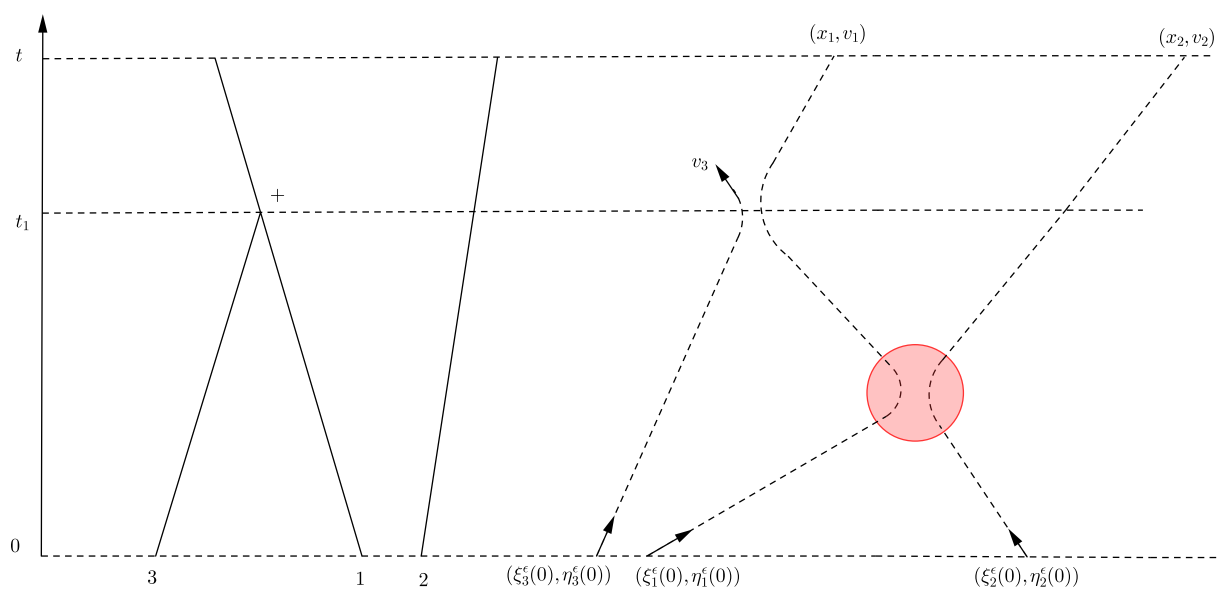

Figure 2.4: On the left, a simple case: $\Gamma(j,n) = \Gamma(2,1) = 1$. The plus sign on the node recalls that $\sigma_1 = +$. We want to estimate the contributions to the corresponding formula $\mathcal{T}_+^\varepsilon(\mathbf{z}_2, t)$, coming from the recolliding trajectories of the IBF. An example of such a trajectory is symbolically represented in the figure on the right.

in outgoing ($\sigma_1 = +$) collision configuration. After the scattering between the couple $(1,3)$, particle 1 collides with particle 2. This is a collision which is not a creation, i.e. what we called a recollision. We shall imagine that $Y = \xi_2^\varepsilon(t_1) - \xi_1^\varepsilon(t_1)$ is order 1 while ε is very small. We neglect the time of scattering between the pair $(1,3)$ and approximate by Y the relative distance between particles 1 and 2 just before the scattering between 1 and 3.

Denote by η_1^- the velocity of particle 1 between time t_1 and the time of the recollision. Then, the recollision implies a geometrical relation between $W = v_2 - \eta_1^-$ and Y. They must be chosen in such a way that there exists $s \in (0, t_1)$ for which $|\xi_2^\varepsilon(s) - \xi_1^\varepsilon(s)| = \varepsilon$. This is implied by the fact that W lies in the cone $C(Y)$ with vertex 0, axis the direction of Y and tangent to the ball of center $-Y$ and radius ε, see Figure 2.5. Moreover, by the laws of

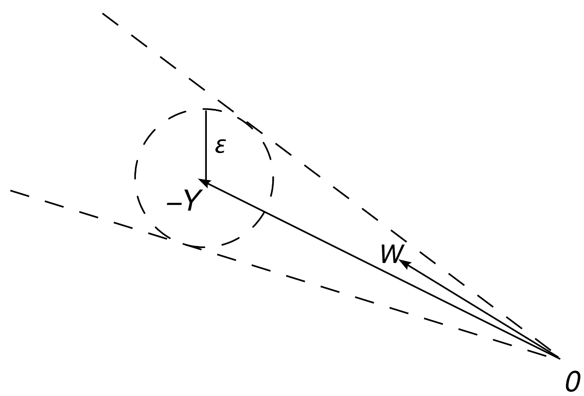

Figure 2.5: The recolllision–cone $C(Y)$.

scattering (2.2.2) and (2.3.24), it is easy to see that η_1^- belongs to the spherical surface of the ball centered in $\frac{v_1+v_3}{2}$ of diameter $\frac{|V|}{2}$, where $V = v_3 - v_1$. In fact, fixed V and v_1 (hence at fixed total momentum), η_1^- moves over that sphere essentially as the scattering vector ω (see Figure 2.1). In conclusion, η_1^- must belong to the intersection A of the cone $v_2 - C(Y)$ and the spherical surface described above. Clearly, at a given $|V|$, the surface measure of A is $O(\varepsilon^2)$ once assumed $Y = O(1)$ and $\varepsilon << 1$.

Now we want to estimate

$$\int_{|v_3|\le R} dv_3 \int_{\nu_1\cdot(v_3-v_1)\ge 0} d\nu_1 \nu_1 \cdot (v_3 - v_1) \mathbb{1}_{\{\eta_1^- \in A\}} \; , \tag{2.8.1}$$

where the cutoff on v_3 has been added here to obtain an integral over a compact set. By the above discussion, it follows that a rather natural way to proceed is to express the integral in terms of an integration with respect to V and ω, so that we get

$$\int d\hat{V} \int_0^{2R} d|V||V|^2 \int d\omega B(\omega, V) \mathbb{1}_{\{\eta_1^- \in A\}} \tag{2.8.2}$$

where $\hat{V}$ is the versor of V, we assumed also $|v_1| \le R$, and B is the function resulting from the change of variables $\nu_1 \to \omega(\nu_1, V)$ (see Eq. (2.2.3)). If B were bounded (as in the case of hard spheres) we would easily conclude that such a contribution is $O(\varepsilon^2)$. Unfortunately, this is not true in many physically interesting cases, since B could not exist as a single–valued function and, even in each monotonicity branch of the scattering map $\rho \to \Theta(\rho)$, it could diverge when the map becomes flat (see the Appendix). Thus to control the integral (2.8.2), we need to know properties of the scattering map (presence and strength of singularities), depending on the details of the potential. In this way it seems difficult to establish a unified analysis of a large class of interactions (see the discussion in the Appendix).

We propose a different method avoiding the use of the scattering cross–section (i.e. of the function B). This is based on two main ideas:

- We work as much as possible on the Boltzmann flow, rather than on the interacting flow. Of course the BBF is much simpler since the interactions (= creations) are instantaneous. Moreover, by virtue of the property described by the Remark on page 49, various parametrizations of the BBF, different from the usual in terms of $\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}$, can be conveniently used. In particular, the trajectories of the BBF can be parametrized by incoming collision variables. For these reasons, we find convenient to estimate the events in which some couple of particles of the BBF get closer than a certain distance (say on a scale slightly larger than ε). Indeed in the complement of this set the Boltzmann trajectories are close to the particle trajectories, as soon as the scattering time is small and the energy is not too large (which will be assured by an additional cutoff).

- To estimate the above set of events, we use as much as possible the integration over time variables. From Figure 2.4 one can guess that, in general, only a small ($O(\varepsilon)$) interval of values of t_1 will be compatible with the recollision condition.

However time integrations may produce singularities for special configurations of relative velocities (see the Remark on page 60). Exploiting the global structure of the BBF, we will prove that such configurations are either excluded by the condition on the “initial datum” $\mathbf{z}_j \in \Omega_j$, or they correspond to small set of values of relative velocities of incoming collisions, which will be estimated using the map $\mathcal{I}^{(r)}$ of Section 2.7.2.

2.8.2 Proof of Theorem 1

By the result in Proposition 2 of Section 2.6 and the reformulations of Section 2.7, the proof of Theorem 1 reduces to the proof of convergence of the generic term of the expansion, i.e.

Proposition 3. *Under Hypotheses 1–4, for all* $\Gamma(j,n), \boldsymbol{\sigma}_n$ *and* $(\mathbf{z}_j, t) \in \Omega_j \times \mathbb{R}^+$,

$$\lim_{\varepsilon \to 0} \mathcal{T}^{\varepsilon}_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t) = \mathcal{T}_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t) \; . \tag{2.8.3}$$

The aim is to apply the dominated convergence theorem to show that the trajectories of the IBF converge almost everywhere to those of the BBF. As already mentioned, we first need to“cut off away” pieces of phase space which correspond to trajectories of the IBF exhibiting recollisions, large scattering times, or high energies, and prove that they give a negligible contribution in the limit. Outside this properly defined set of "bad events", we will be able to estimate explicitly the distance between the interacting and the Boltzmann trajectories.

In all this section and in the following we will keep fixed $\mathbf{z}_j \in \Omega_j$ and $t > 0$. Moreover the times $\mathbf{t}_n$ will always be supposed to be ordered (see (2.7.12)), and the $\boldsymbol{\nu}_n$ to satisfy the constraint implied by $\boldsymbol{\sigma}_n$ (Eq. (2.7.16)). In the present section we also fix $\Gamma(j,n)$ and $\boldsymbol{\sigma}_n$.

We start by focusing on the BBF $\boldsymbol{\zeta}(s)$ giving a new definition. Consider particle i and look at the graph of $\Gamma(j,n)$. A polygonal path $\mathcal{P}_i$ is uniquely defined if we walk on the tree by going forward in time, starting from the time–zero endpoint of line i and going up to the root–point at time t. See for instance Figure 2.6. To $\mathcal{P}_i$ we may naturally associate a one–

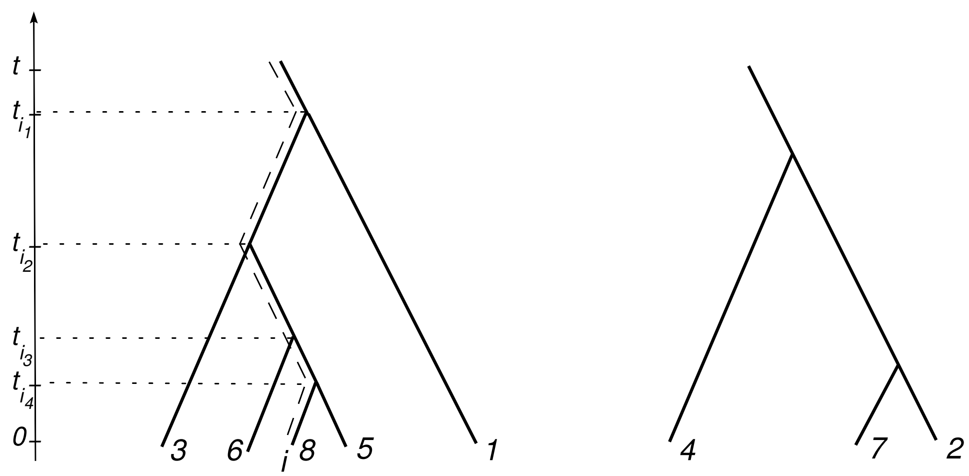

Figure 2.6: The line closest to the dashed line is the path $\mathcal{P}_i$ in the tree $\Gamma(2,6)$, with $i = 8$. The states of the particle associated to it via the BBF form the"virtual trajectory".

particle piecewise–free trajectory, built up with pieces of trajectories of (different) particles of the BBF. More precisely, fixed a BBF with parameters $(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n})$, denote $t_{i_1}, \cdots, t_{i_{n_i}}$ the (decreasing) subsequence of $t_1, \cdots, t_n$ of the times corresponding to the nodes met by following the path $\mathcal{P}_i$ (n_i being the number of such nodes, with the convention $i_0 = 0, t_{i_0} = t$): see the figure. We call *virtual trajectory associated to particle i in the BBF*, and indicate it by $\zeta^i(s) = (\xi^i(s), \eta^i(s)) \in \mathbb{R}^6$ with $s \in [0,t]$, the one–particle trajectory given by:

$$\zeta^i(s) = \begin{cases} \zeta_i(s) & \text{for } s \in [0, t_{i_{n_i}}) \\ \zeta_{k_{i_r}}(s) & \text{for } s \in [t_{i_r}, t_{i_{r-1}}), \quad 0 < r \le n_i \end{cases} . \tag{2.8.4}$$

Observe that, during the time of existence of particle i in the BBF, $\zeta^i(s) = \zeta_i(s)$.

Now consider a couple of particles (i,h) and compare their virtual trajectories. Calling"root"of $\mathcal{P}_i$ the root line of the tree to which $\mathcal{P}_i$ belongs, we have two possibilities: either the roots of $\mathcal{P}_i$ and $\mathcal{P}_h$ coincide (i.e. i and h belong to the same single tree), or not. In the first case, there exists (uniquely) a node of the tree where $\mathcal{P}_i$ and $\mathcal{P}_h$ merge. For any given couple (i, h) we introduce the subsequence of $t_1, \cdots, t_n$:

$$t \geq t^0 > t^1 > t^2 > \cdots > t^{n_{ih}} > t^{n_{ih}+1} \equiv 0 \ , \tag{2.8.5}$$

defined as follows. Time t^0 is equal to t if $\mathcal{P}_i$ and $\mathcal{P}_h$ have different roots; otherwise, it is equal to the time (strictly smaller than t) of the node where $\mathcal{P}_i$ and $\mathcal{P}_h$ merge. The sequence $t^1, \cdots, t^{n_{ih}}$ is given by the ordered union of the times $t_{i_1}, \cdots, t_{i_{n_i}}$ and $t_{h_1}, \cdots, t_{h_{n_h}}$ that belong to the interval $(0, t^0)$. Here n_{ih} is the number of such times, and $t^{n_{ih}+1}$ has been put equal to zero by convention. See also Figure 2.7 below.

We are ready to define a part of the"bad set"to be cutoffed. Let be $\delta > 0$. The *set of $\delta-$overlaps*, $\mathcal{N}(\delta) \subset \times\mathbb{R}^n \times S^{2n} \times \mathbb{R}^{3n}$, is

$$\mathcal{N}(\delta) = \left\{ \mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n} \ \Big| \ \min_{i<h} \ \min_{s \in [0,t^1]} |\xi^i(s) - \xi^h(s)| \leq \delta \right\} , \tag{2.8.6}$$

where t^1 depends on the couple (i, h) under consideration. Notice that this set is completely defined via the BBF. Clearly, it depends also on $\mathbf{z}_j, t$.

Note also that the set $\mathcal{N}(\delta)$ detects the δ–overlaps (namely when $|\xi^i(s) - \xi^h(s)| \leq \delta$) of the virtual paths $\mathcal{P}_i$ and $\mathcal{P}_h$ excluding the time interval $(t^1, t^0]$.

In the following, $\delta > \varepsilon$ is a function of ε going to zero as $\varepsilon \to 0$. Then, the first step in the proof is to show that the restriction of the integrals contained in $\mathcal{T}^{\varepsilon}_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t)$ to the set $\mathcal{N}(\delta)$ is arbitrarily small with ε. To do so, consider the *set of point–overlaps* $\mathcal{N} \equiv \mathcal{N}(0)$,

$$\mathcal{N} = \left\{ \mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n} \ \Big| \ \min_{i<h} \ \min_{s \in [0,t^1]} |\xi^i(s) - \xi^h(s)| = 0 \right\} . \tag{2.8.7}$$

Obviously it is

$$\lim_{\delta \to 0} \mathbb{1}_{\mathcal{N}(\delta)} = \mathbb{1}_{\mathcal{N}} \ . \tag{2.8.8}$$

We will now show that the set $\mathcal{N}$ has $d\Lambda-$measure zero. Precisely, we will show that the condition in (2.8.7) implies a certain number of relations between the integration variables that can be satisfied at most for a $d\Lambda-$null set of values.

If we are in $\mathcal{N}$, then for some couple (i, h) there exists

$$t^* = \max\{s \in [0, t^1] \text{ s.t. } |\xi^i(s) - \xi^h(s)| = 0\} \ . \tag{2.8.9}$$

It will be $t^* \in [t^{l+1}, t^l)$ for some $l \in \{0, \cdots, n_{ih}\}$. l is the total number of interactions in the virtual trajectories of i and h between the overlapping time and t^0. For $q = 0, \cdots, l$, we define:

$$\begin{aligned} &Y^q = \xi^h(t^q) - \xi^i(t^q) \ , \\ &\eta^i_q \equiv \eta^i(s) \ , \ \eta^h_q \equiv \eta^h(s) \qquad \text{for} \ \ s \in (t^{q+1}, t^q) \ , \\ &W^q = \eta^h_q - \eta^i_q \ . \end{aligned} \tag{2.8.10}$$

We indicate by $f \in \{0, 1, \cdots, n\}$ the index such that

$$t^0 = t_f \ . \tag{2.8.11}$$

Notice that either $t^0 = t$ $(f = 0)$ and $Y^0 \neq 0$ (because $\mathbf{z}_j \in \Omega_j$) or $t^0 < t$ $(f > 0)$ and $Y^0 = 0$ (because t^0 is the time of the node where $\mathcal{P}_i$ and $\mathcal{P}_h$ merge). A possible event in $\mathcal{N}(\delta)$ is pictured in Figure 2.7.

Given a point in $\mathcal{N}$, we observe preliminarily that we may assume:
(i) if $Y^0 = 0$, then $W^0 \neq 0$;
(ii) $l \geq 1$;
(iii) $W^l \neq 0$.
Assumption (i) corresponds to exclude subsets of the integration domain of codimension 3. In fact, by the laws of the two–body scattering, if $Y^0 = 0$ then $W^0 \neq 0$ except for a single value of the velocity of the particle created at time $t^0 = t_f$, namely v_{j+f} must be equal to $\eta_{k_f}(t_f^+)$. Note that $l = 0$ and $Y^0 \neq 0$ is impossible because $\mathbf{z}_j \in \Omega_j$. On the other hand if $l = 0$ and $Y^0 = 0$, then necessarily $W^0 = 0$, which we excluded by (i). Finally if $W^l = 0$, then the overlap takes place in the interval $[t^l, t^{l-1})$ and this contradicts the definition of l.

The point–recollision condition is verified only if $\min_s |Y^l - W^l s| = 0$, which in turn implies $Y^l \wedge \hat{W}^l = 0$, where $\hat{W}^l = \frac{W^l}{|W^l|}$. Since

$$Y^l = Y^0 - \sum_{q=0}^{l-1} W^q (t^q - t^{q+1}) \ , \tag{2.8.12}$$

we have

$$\begin{aligned} 0 &= Y^0 \wedge \hat{W}^l - \sum_{q=0}^{l-1} (W^q \wedge \hat{W}^l)(t^q - t^{q+1}) \\ &= (Y^0 - W^0 t^0) \wedge \hat{W}^l - \sum_{q=1}^{l} [(W^q - W^{q-1}) \wedge \hat{W}^l)] t^q \ . \end{aligned} \tag{2.8.13}$$

But, by the Remark on page 49, all the vectors involved in this relation do not depend on time. Hence as soon as $[(W^q - W^{q-1}) \wedge \hat{W}^l)] \neq 0$ for some q, there exists at most one value of the time t^q fulfilling condition (2.8.13).

Otherwise, it will be

$$\hat{W}^l \wedge W^q = 0 \qquad \text{for all } q = 0, 1, \cdots, l \ , \tag{2.8.14}$$

i.e. all (not vanishing) relative velocities are collinear. In particular, Eq. (2.8.13) implies

$$Y^0 \wedge \hat{W}^l = 0 \ . \tag{2.8.15}$$

As said above we have two cases, which we treat separately.

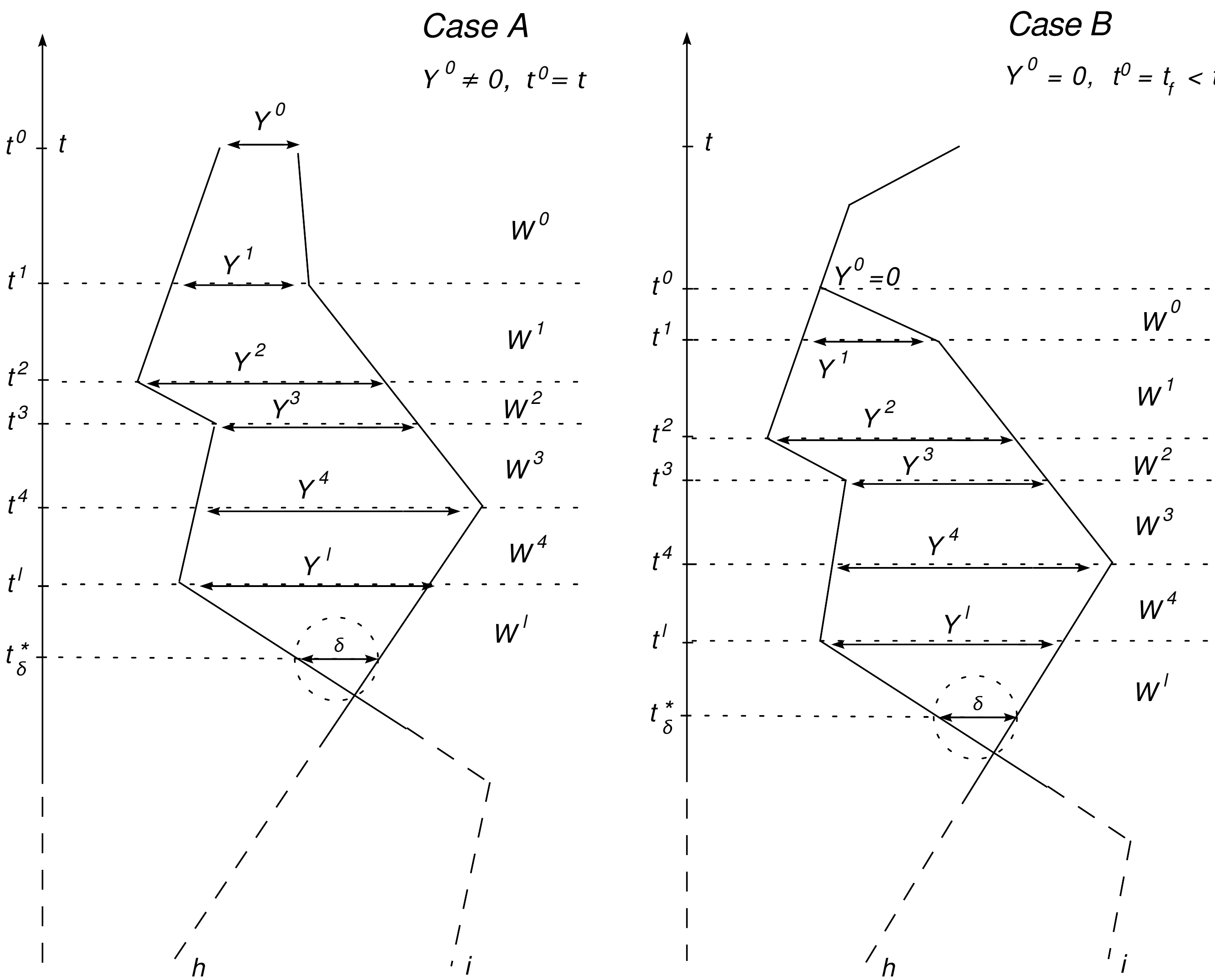

Figure 2.7: A symbolical drawing of virtual trajectories of two particles i, h in the BBF showing a $\delta-$overlap after $l = 5$ interactions. Relative distances and velocities are indicated as defined in (2.8.10). In case A the two particles belong to different trees, while in case B they belong to the same tree. t^*_δ is the first (backwards) time when the particles reach the distance δ.

- *Case* $Y^0 \neq 0, t^0 = t$.

  Both Y^0 and W^0 are collinear with $\hat{W}^l$. Therefore

$$Y^0 \wedge W^0 = 0 \;, \tag{2.8.16}$$

  which is excluded since $\mathbf{z}_j \in \Omega_j$.

- *Case* $Y^0 = 0, t^0 < t$.

  If all relative velocities W^q are coincident, then no point–recollision is possible since $W^0 \neq 0$. Finally assume that $U := W^{\overline{q}} - W^{\overline{q}-1} \neq 0$ for some $\overline{q} \in \{1, \cdots, l\}$ and put $\hat{U} = \frac{U}{|U|}$. We have $\hat{U} \wedge \hat{W}^l = 0$, which, together with $W^0 \wedge \hat{W}^l = 0$, implies

$$\hat{U} \wedge W^0 = 0 \;. \tag{2.8.17}$$

  Let $t^{\overline{q}} = t_{f'}$, where $f' > f$ (recall (2.8.11)). We change variables

$$(\nu_f, v_{j+f}, \nu_{f'}, v_{j+f'}) \to (\nu'_f, V'_f, \nu_{f'}, V_{f'}) \tag{2.8.18}$$

  according to

$$\begin{cases} (\boldsymbol{\nu}_{f-1}, \nu'_f, \boldsymbol{\nu}_{f,n-f}, \mathbf{v}_{j,f-1}, V'_f, \mathbf{v}_{j+f,n-f}) = \mathcal{I}^{(f)}(\boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \\ V_{f'} = v_{j+f'} - \eta_{k_{f'}}(t^+_{f'}) \end{cases} . \tag{2.8.19}$$

  In the first equation we used the map on page 50, while the second is a simple translation. We have $W_0 = V'_f$ and $U = \pm[\eta_{k_{f'}}(t^+_{f'}) - \eta_{k_{f'}}(t^-_{f'})]$ or $U = \pm[\eta_{k_{f'}}(t^+_{f'}) - \eta_{j+f'}(t^-_{f'})]$; see Figure 2.8. From the rules of scattering it follows that the vector U depends only on $(\nu_{f'}, V_{f'})$[2]. Then Eq. (2.8.17) defines a subset of codimension two in the space of variables $(V'_f, \nu_{f'}, V_{f'})$. By Lemma 4 of Section 2.7.2, the change of variables (2.8.18)–(2.8.19) is a one–to–one measure preserving map. Therefore the subset defined by (2.8.17) has $d\Lambda-$measure zero.

So far we have proven that $\mathcal{N}$ is a null set and then it follows that the integral of any finite measure restricted to $\mathcal{N}(\delta)$ goes to zero with δ. But, in our Hypothesis 3, $d\Lambda \left(\prod B^\varepsilon\right) f^N_{0,j+n}$ is uniformly bounded by a finite measure for ε small. Indeed, using conservation of energy, we can estimate it by

$$d\Lambda \left(2\sqrt{\sum_{i=1}^{j+n} v_i^2} \right)^n e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \|f^N_{0,j+n}\|_\beta \;, \tag{2.8.20}$$

where we used that the energy of the IBF at time t is purely kinetic if ε is small enough (having fixed $\mathbf{x}_j$ outside the diagonals). Hence we have what we asserted before Eq. (2.8.7), that is

$$\lim_{\varepsilon \to 0} \int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \prod_{i=1}^n B^\varepsilon(\nu_i; v_{j+i} - \eta^\varepsilon_{k_i}(t_i)) \mathbb{1}_{\mathcal{N}(\delta)} f^N_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)) = 0 \tag{2.8.21}$$

[2]It can be $U = \pm\omega(\omega \cdot V_{f'})$ or $U = \pm[V_{f'} - \omega(\omega \cdot V_{f'})]$ where $\omega = \omega(\nu_{f'}, V_{f'})$ is the scattering vector.

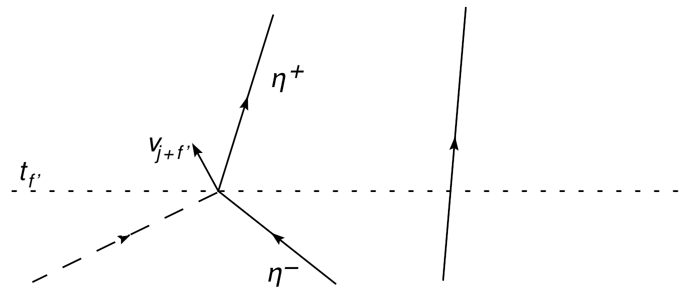

Figure 2.8: Detail of the virtual trajectories of Figure 2.7–Case B, for times close to $t^{\bar{q}} = t_{f'}$. In the example $\sigma_{f'} = +$. The difference of relative velocities is $U = W^{\bar{q}} - W^{\bar{q}-1} = \eta^+ - \eta^-$ (here f' belongs to $\mathcal{P}_i$; if f' belongs to $\mathcal{P}_h$ then $U = \eta^- - \eta^+$), where $\eta^+ = \eta_{k_{f'}}(t^+_{f'})$ and η^- can be equal to $\eta_{j+f'}(t^-_{f'})$ or $\eta_{k_{f'}}(t^-_{f'})$, depending on the structure of $\mathcal{P}_i$. The variables describing the scattering are $(\nu_{f'}, v_{j+f'})$, or alternatively $(\nu'_{f'}, V'_{f'})$.

for all $\mathbf{z}_j \in \Omega_j$ and all $t > 0$.

Remark. In the above proof we try to use time variables, when possible, to show that the overlaps are rare. Otherwise we have to analyze geometrical conditions involving relative velocities of the virtual trajectories, namely Equation (2.8.14), which is proven to be a vanishing measure condition either by integrating in the incoming relative velocities of a node of the tree (case $Y^0 = 0$), or by showing that the condition is impossible by definition of Ω_j (case $Y^0 \neq 0$).

In the following section, when we will deal with quantitative estimates of the recollision (or overlap) events, we will follow the same strategy, that is we will integrate in time unless we face the condition "$|\hat{W}^l \wedge W^q|$ small for all q". Note that, in particular, this is the case of a sequence of central and grazing collisions in the virtual trajectories, for which the relative velocities may remain unchanged (remind that the scattering cross–section may possibly have concentrations on such collisions: see the Appendix). On the other hand, in this case the virtual trajectories are analogous to a free flow for which the recollisions can be easily controlled.

Besides $\mathcal{N}(\delta)$, we still have to take care of some additional subsets of the integration region. Let $\mu \in (0,1)$. Putting

$$\begin{aligned} \mathbb{1}^\varepsilon_1 &= \mathbb{1}_{\{\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2 < |\log\varepsilon|\}} \,, \\ \mathbb{1}^\varepsilon_2 &= \prod_{r=1}^{n} \mathbb{1}_{\{|(v_{j+r} - \eta^\varepsilon_{k_r}(t_r)) \wedge \nu_r| > \varepsilon^\mu\}} \,, \end{aligned} \tag{2.8.22}$$

a simple estimate as the one in (2.8.20) is sufficient to show that

$$\lim_{\varepsilon\to 0} \int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \prod_{i=1}^{n} B^\varepsilon(\nu_i; v_{j+i} - \eta^\varepsilon_{k_i}(t_i))(1 - \mathbb{1}_{\mathcal{N}(\delta)})(1 - \mathbb{1}^\varepsilon_1 \mathbb{1}^\varepsilon_2) f^N_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)) = 0 \,. \tag{2.8.23}$$

Thus, to obtain the final result we are left with the proof of

$$\lim_{\varepsilon\to 0}\int d\Lambda(\mathbf{t}_n,\boldsymbol{\nu}_n,\mathbf{v}_{j,n})\prod_{i=1}^{n} B^{\varepsilon}(\nu_i;v_{j+i}-\eta^{\varepsilon}_{k_i}(t_i))(1-\mathbb{1}_{\mathcal{N}(\delta)})\mathbb{1}^{\varepsilon}_1\mathbb{1}^{\varepsilon}_2 f^N_{0,j+n}(\boldsymbol{\zeta}^{\varepsilon}(0))=\mathcal{T}_{\boldsymbol{\sigma}_n}(\mathbf{z}_j,t)\ . \tag{2.8.24}$$

Notice that up to now we did not use any property of the interacting flow but the conservation of energy. Now we have to examine in more detail the structure of the IBF and to compare it with the Boltzmann flow. Since we work in the complement of $\mathcal{N}(\delta)$ and in the sets in (2.8.22), we are actually in a favorable situation in proving that the distance between the two flows is small. Indeed, as we will show in the following lemma, the IBF has no recollisions and its differences with the BBF are only due to the scattering time, which is absent in the Boltzmann flow, and to the $\varepsilon-$delocalization of the created particles (also absent in the BBF).

Choose

$$\delta=\varepsilon^{1-\mu}(\log\varepsilon)^2\ . \tag{2.8.25}$$

Then we have:

Lemma 5. *If $(\mathbf{t}_n,\boldsymbol{\nu}_n,\mathbf{v}_{j,n})$ is outside $\mathcal{N}(\delta)$ and inside the sets* (2.8.22)*, then for ε sufficiently small and all $i=1,\cdots,j+n$*

$$\max_{s\in[0,t]}|\xi_i(s)-\xi^{\varepsilon}_i(s)|\le D\varepsilon^{1-\mu}|\log\varepsilon|^{\frac{3}{2}} \tag{2.8.26}$$

for some $D>0$. In particular, the IBF does not admit recollisions. Moreover, $\eta^{\varepsilon}_{k_r}(t_r)=\eta_{k_r}(t_r^+)$ for all $r=1,\cdots,n$ and $\eta^{\varepsilon}_i(0)=\eta_i(0)$.

Proof of Lemma 5. The proof is based on a simple continuity argument. We proceed by induction on r proving that, for some $D'>0$,

$$|\xi_i(s)-\xi^{\varepsilon}_i(s)|\le D'r\varepsilon^{1-\mu}|\log\varepsilon|^{\frac{1}{2}}\ ,\qquad s\in(t_{r+1},t_r) \tag{2.8.27}$$

for $i=1,\cdots,j+r$, from which (2.8.26) follows since n is smaller then a constant times $|\log\varepsilon|$ for ε small. A byproduct will be that if particle $i\le j+r-1$, at time t_r, has already completed the (possible) scattering with its progenitor (or its last son) in the IBF, then necessarily $\eta^{\varepsilon}_i(t_r)=\eta_i(t_r)$ (from which the last assertion of the lemma follows, because the parameters are taken outside $\mathcal{N}(\delta)$).

For $r=0$ the statement is trivial. Indeed, since we are outside $\mathcal{N}(\delta)$ with $\delta>\varepsilon$ and the states at time t of the BBF and the IBF are the same then, for $s\in(t_1,t)$, $\zeta_i(s)=\zeta^{\varepsilon}_i(s)$.

Let now $s\in(t_{r+1},t_r)$ for $r>0$ and $i=1,\cdots,j+r$. We consider the case (being the other cases easier) in which particle i in the IBF is interacting at time t_r with another particle h. For ε small and by inductive hypothesis, this can be true only if the two particles coincide with the couple $(k_r,j+r)$, or if they are a couple progenitor–son, $(k_{r'},j+r')$ with $r'<r$, that has not finished the two–body scattering. Also, note that no other particle can be at distance $\le\delta/2$ from i and h at time t_r (for ε small and $n=O(|\log\varepsilon|))$.

Denote by $s^* \in (t_{r+1}, t_r)$ the last (first backwards) recollision time of particle i or h in the IBF, that is $|\xi^\varepsilon_{h'}(s) - \xi^\varepsilon_i(s)| > \varepsilon$ for all $h' \neq h, i$ and all $s \in (s^*, t_r)$, and same equation with i replaced by h. Then, for the same values of s, particles i and h behave as they were isolated and we have that $|\xi^\varepsilon_i(s) - \xi_i(s)|$ is equal to the same quantity evaluated in $\max(s, t_r - \bar{t})$, where $\bar{t}$ is the time of the two–body scattering (in fact, once the backward scattering finishes, it must be $\eta^\varepsilon_i(s) = \eta_i(s)$). Hence, taking into account Lemma 1 and (2.8.22), we have

$$\begin{aligned} |\xi^\varepsilon_i(s) - \xi_i(s)| &\leq |\xi^\varepsilon_i(t_r) - \xi_i(t_r)| + 2\sqrt{2/\beta |\log \varepsilon|}\,\bar{t} \\ &\leq D'(r-1)\varepsilon^{1-\mu} |\log \varepsilon|^{\frac{1}{2}} + 2\sqrt{2/\beta |\log \varepsilon|} A \varepsilon^{1-\mu} \ , \end{aligned} \tag{2.8.28}$$

which implies (2.8.27) with $D' = 2\sqrt{2/\beta} A$, for $s \in (s^*, t_r)$. The same holds of course for particle h.

In particular (taking ε small so that n is $O(|\log \varepsilon|)$) we have that $|\xi^\varepsilon_i(s) - \xi_i(s)| \leq \frac{\delta}{4}$ up to the first recollision time. Since we are outside $\mathcal{N}(\delta)$, for all $h' \neq h, i$

$$\begin{aligned} |\xi^\varepsilon_{h'}(s) - \xi^\varepsilon_i(s)| &\geq |\xi_{h'}(s) - \xi_i(s)| - |\xi^\varepsilon_{h'}(s) - \xi_{h'}(s)| - |\xi^\varepsilon_i(s) - \xi_i(s)| \\ &> \delta - \frac{\delta}{4} - \frac{\delta}{4} = \frac{\delta}{2} \end{aligned} \tag{2.8.29}$$

up to the first recollision time of i, h, h', and the same equation holds for particle h. But $\delta/2 > \varepsilon$. Therefore, by continuity of the flow, $|\xi^\varepsilon_{h'}(s) - \xi^\varepsilon_i(s)| > \frac{\delta}{2}$ and $|\xi^\varepsilon_{h'}(s) - \xi^\varepsilon_h(s)| > \frac{\delta}{2}$ for all times $s \in (t_{r+1}, t_r)$ and Eq. (2.8.27) holds in the full interval. □

To conclude the proof of Proposition 3, we note that the above result, together with Hypothesis 4, can be also used to replace the initial datum $f^N_{0,j+n}$ by $f_{0,j+n}$ in $\mathcal{T}^\varepsilon_{\boldsymbol{\sigma}_n}(\mathbf{z}_j, t)$, that is to show that

$$\lim_{\varepsilon \to 0} \int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \prod_{i=1}^n B^\varepsilon(\nu_i; v_{j+i} - \eta^\varepsilon_{k_i}(t_i)) \left| f^N_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)) - f_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)) \right| = 0 \ . \tag{2.8.30}$$

Indeed we can restrict the above integrals for values of $\boldsymbol{\zeta}^\varepsilon(0)$ in a compact subset of $\mathcal{M}_{j+n}$, producing an arbitrarily small error.

Using this last result, estimate (2.8.20) (with $f^N_{0,j+n}$ replaced by $f_{0,j+n}$), dominated convergence theorem and Lemma 5, Eq. (2.8.24) follows. □

2.8.3 Proof of Theorem 2

In this section we go through the steps in the proof of Theorem 1, estimating explicitly all the error terms arising in the limiting procedure, under the additional Hypothesis 5. We also need to take care of the dependence on n to guarantee the summability. For instance the bound (2.8.20) is useless in the present context because it grows as $(n!)^{1/2}$.

In this section, for notational simplicity, we will denote by C all pure positive constants depending only on β and α. We also denote by $\mathcal{E}_1, \mathcal{E}_2, \cdots$ all the errors in our procedure (i.e. it will be $f_j^N(\mathbf{z}_j,t) - f_j(\mathbf{z}_j,t) = \sum_i \mathcal{E}_i$).

We start again noticing that it is enough to control the difference

$$|\tilde{f}_j^N(\mathbf{z}_j,t) - f_j(\mathbf{z}_j,t)| \ , \tag{2.8.31}$$

since, by Proposition 2, the error due to this approximation is

$$|\mathcal{E}_1| = |f_j^N - \tilde{f}_j^N| \leq C^j \varepsilon \tag{2.8.32}$$

for t sufficiently small.

Moreover, since $1 - \alpha_n^\varepsilon(j) \leq jC^n\varepsilon^2$ for $n \leq N-j$, we have also

$$|\mathcal{E}_2| = \sum_{n\geq 0}^{N-j} (1 - \alpha_n^\varepsilon(j)) \Big| \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} (-1)^{\boldsymbol{\sigma}_n} \mathcal{T}_{\boldsymbol{\sigma}_n}^\varepsilon \Big| \leq C^j \varepsilon^2 \ . \tag{2.8.33}$$

A third simple error arises by neglecting the rest of the series expansion so that we focus on

$$\sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} |\mathcal{T}_{\boldsymbol{\sigma}_n}^\varepsilon - \mathcal{T}_{\boldsymbol{\sigma}_n}| \ , \tag{2.8.34}$$

where

$$\overline{n} = D''|\log \varepsilon| \tag{2.8.35}$$

with ε small enough. The error generated by this last cutoff is bounded by the remainder of the geometric series appearing in formula (2.6.10), therefore if we choose $D'' = |\log(tC'_{\beta,\alpha})|^{-1}$ it is

$$|\mathcal{E}_3| \leq \sum_{n\geq\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} \left(\mathcal{T}_{\boldsymbol{\sigma}_n}^\varepsilon + \mathcal{T}_{\boldsymbol{\sigma}_n} \right) \leq C^j \varepsilon \ , \tag{2.8.36}$$

for t sufficiently small.

Before considering the difference (2.8.34), we estimate the error caused by the restriction of the integrals to suitable sets of integration variables, where the convergence of the flows (Lemma 5) is guaranteed.

However, first of all, to simplify the expression of the integrands avoiding estimate (2.8.20), we get rid of $(\prod B^\varepsilon)$ performing the decomposition $\{(\prod B^\varepsilon) > \varepsilon^{-\lambda}\}$ and $\{(\prod B^\varepsilon) \leq \varepsilon^{-\lambda}\}$ (and the same for $(\prod B)$) for a suitable $\lambda \in (0,1)$. Moreover we shall often use in the following the bounds

$$\begin{aligned} &f_{0,j+n}^N(\boldsymbol{\zeta}^\varepsilon(0)) \mathbb{1}_{\{|\xi_{j+i}^\varepsilon(t_i) - \xi_k^\varepsilon(t_i)| > \varepsilon \ \forall k \neq k_i\}} \leq C^{j+n} e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \ , \\ &f_{0,j+n}(\boldsymbol{\zeta}(0)) \leq C^{j+n} e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \ , \end{aligned} \tag{2.8.37}$$

consequences of Hypotheses 2 and 3, the conservation of energy and the fact that the energy at time t is purely kinetic if $\mathbf{x}_j$ is outside the diagonals and ε is small enough.

We need to estimate

$$
\begin{aligned}
|\mathcal{E}_4| \le & \sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} \Big[\int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \mathbb{1}_{\{(\prod B^\varepsilon) > \varepsilon^{-\lambda}\}} \prod_{i=1}^n B^\varepsilon(\nu_i; v_{j+i} - \eta^\varepsilon_{k_i}(t_i)) f^N_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)) \\
& + \int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \mathbb{1}_{\{(\prod B) > \varepsilon^{-\lambda}\}} \prod_{i=1}^n B(\nu_i; v_{j+i} - \eta_{k_i}(t_i^+)) f_{0,j+n}(\boldsymbol{\zeta}(0)) \Big] \\
& \le \varepsilon^\lambda \sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} C^{j+n} \int d\Lambda \; e^{-\frac{\beta}{2} \sum_{i=1}^{j+n} v_i^2} \Big[\Big(\prod B^\varepsilon \Big)^2 + \Big(\prod B \Big)^2 \Big] \, .
\end{aligned}
\tag{2.8.38}
$$

Remind now the expression of B^ε and Eq. (2.7.4). Since $\sum_{k_i=1}^{j+i-1} \left(\eta^\varepsilon_{k_i}(t_i) \right)^2$ is bounded by the total energy $\sum_{i=1}^{j+n} v_i^2$ (being the potential positive), it follows easily that

$$
\sum_{\Gamma(j,n)} \Big(\prod B^\varepsilon \Big)^2 \le 2^n \prod_{i=1}^n \left((j+n) v_{j+i}^2 + \sum_{l=1}^{j+n} v_l^2 \right) \, . \tag{2.8.39}
$$

The same estimate holds for $\sum_{\Gamma(j,n)} \left(\prod B \right)^2$. Therefore

$$
|\mathcal{E}_4| \le \varepsilon^\lambda \sum_{n \ge 0} C^{j+n} \int d\Lambda \prod_{i=1}^n \left((j+n) v_{j+i}^2 e^{-\frac{\beta}{4} v_{j+i}^2} + \frac{4n}{e\beta} e^{-\frac{\beta}{4} v_{j+i}^2} \right) \tag{2.8.40}
$$

where we used the bound

$$
\sum_{i=1}^{j+n} v_i^2 e^{-\frac{\beta}{4n} \sum_{i=1}^{j+n} v_i^2} \le \frac{4n}{e\beta} \, . \tag{2.8.41}
$$

The integral on the velocities in (2.8.40) factorizes so that

$$
|\mathcal{E}_4| \le \varepsilon^\lambda \sum_{n \ge 0} C^{j+n} \frac{t^n}{n!} (j+n)^n \, . \tag{2.8.42}
$$

Since

$$
\frac{(j+n)^n}{n!} \le \frac{(j+n)^{j+n}}{(j+n)!} \le e^{j+n} \, , \tag{2.8.43}
$$

we have that (2.8.42) is bounded by a geometric series. Hence we conclude that

$$
|\mathcal{E}_4| \le C^j \varepsilon^\lambda \tag{2.8.44}
$$

for t sufficiently small.

At this point we just follow the lines of the proof of Proposition 3. Using the notations introduced on pages 54–56 to define the set of $\delta-$overlaps in the BBF, we want to estimate

$$
\begin{aligned}
|\mathcal{E}_5| \le & \sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} \Big[\int d\Lambda \; \mathbb{1}_{\{(\prod B^\varepsilon) \le \varepsilon^{-\lambda}\}} \Big(\prod B^\varepsilon \Big) \mathbb{1}_{\mathcal{N}(\delta)} f^N_{0,j+n} \\
& + \int d\Lambda \; \mathbb{1}_{\{(\prod B) \le \varepsilon^{-\lambda}\}} \Big(\prod B \Big) \mathbb{1}_{\mathcal{N}(\delta)} f_{0,j+n} \Big] \, .
\end{aligned}
\tag{2.8.45}
$$

By (2.8.37) it is

$$|\mathcal{E}_5| \leq \varepsilon^{-\lambda} \sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} C^{j+n} \int d\Lambda \; \mathbb{1}_{\mathcal{N}(\delta)} e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \; . \tag{2.8.46}$$

We have the following crucial result:

Lemma 6. *Let $\delta = \varepsilon^{1-\mu}(\log\varepsilon)^2$ (see Eq. (2.8.25)). Given $\mathbf{z}_j \in \Omega_j$, if ε is sufficiently small and $1 \leq n \leq \overline{n}$,*

$$\int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \mathbb{1}_{\mathcal{N}(\delta)} e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \leq C^{j+n} \frac{t^n}{n!} \delta^{\frac{2}{5}} |\log\varepsilon|^{\frac{7}{2}} \; . \tag{2.8.47}$$

Here the choice of δ is not strict and is determined by Lemma 5.

Proof of Lemma 6. First notice that, given a point in $\mathcal{N}(\delta)$, there exists

$$t_\delta^* = \max\{s \in [0, t^1] \text{ s.t. } |\xi^i(s) - \xi^h(s)| \leq \delta\} \tag{2.8.48}$$

for some couple of particles (i,h). Substituting t^* by t_δ^*, we may define l, Y^q, W^q, f as after (2.8.9); see also Figure 2.7.

The case $l = 0$ and $t^0 = t, Y^0 \neq 0$ is made impossible by $\mathbf{z}_j \in \Omega_j$, as soon as δ (i.e. ε) is smaller than a constant depending on $\mathbf{z}_j$. Conversely, the case $l = 0$ and $t^0 < t, Y^0 = 0$ occurs whenever a creation in the BBF is such that the two particles progenitor–son do not separate enough before their next (backwards) interaction. In formulas, $|W^0|(t^0 - t^1) \leq \delta$. This case is controlled by introducing

$$\mathbb{1}_0^\delta = \prod_{r=1}^{n} \mathbb{1}_{\{|(v_{j+r} - \eta_{k_r}(t_r^+))|(t_r - t_{r+1}) > 2t\delta^{2/5}\}} \tag{2.8.49}$$

(remember that the modulus of relative velocity is conserved at collisions). Clearly it is

$$\mathbb{1}_{\mathcal{N}(\delta)} \leq \mathbb{1}_0^\delta \mathbb{1}_{\mathcal{N}(\delta)} + \sum_{r=1}^{n} \mathbb{1}_{\{|(v_{j+r} - \eta_{k_r}(t_r^+))|(t_r - t_{r+1}) \leq 2t\delta^{2/5}\}} \; . \tag{2.8.50}$$

The reason for the choice of the threshold in definition (2.8.49) will be clear soon. For the moment note that we have

$$\sum_{r=1}^{n} \int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \mathbb{1}_{\{|(v_{j+r} - \eta_{k_r}(t_r^+))|(t_r - t_{r+1}) \leq 2t\delta^{2/5}\}} e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \leq C^n \frac{t^{n-1}}{(n-1)!} \delta^{\frac{2}{5}} \; . \tag{2.8.51}$$

In fact, performing first the integration in $dt_r dv_{j+r}$, reminding that $\eta_{k_r}(t_r^+) = \eta_{k_r}(t_{r-1}^-)$ is independent on t_r, v_{j+r}, setting $s = t_r - t_{r+1}$ and $V = v_{j+r} - \eta_{k_r}(t_r^+)$, we find

$$\begin{aligned} &\int_{t_{r+1}}^{t_{r-1}} dt_r \int dv_{j+r} \mathbb{1}_{\{|(v_{j+r} - \eta_{k_r}(t_r^+))|(t_r - t_{r+1}) \leq 2t\delta^{2/5}\}} e^{-\frac{\beta}{2} v_{j+r}^2} \\ &\leq \int_0^{\delta^{2/5}} ds \int dv_{j+r} e^{-\frac{\beta}{2} v_{j+r}^2} + \int_{\delta^{2/5}}^{t} ds \int_0^{\frac{2t\delta^{2/5}}{s}} d|V| 4\pi |V|^2 \\ &\leq C\delta^{\frac{2}{5}} \; . \end{aligned} \tag{2.8.52}$$

Since the integrals in the other variables give $C^{n-1}t^{n-1}/n!$, we get (2.8.51).

Suppose now that the considered point in $\mathcal{N}(\delta)$ is such that $\mathbb{1}_0^\delta \mathbb{1}_{\mathcal{N}(\delta)} = 1$. The $\delta-$overlap condition is verified only if

$$\min_s |Y^l - W^l s| \leq \delta \tag{2.8.53}$$

for some $s \in [0, t^l)$, with Y^l given by (2.8.12). It must be $l \geq 1$ and $W^l \neq 0$. Moreover, the relative velocities W^q cannot be all too close to each other, i.e. the two characteristic functions imply

$$\sum_{q=1}^{l} |W^q - W^{q-1}| > \delta^{\frac{2}{5}} \ . \tag{2.8.54}$$

Otherwise it would be $|W^q - W^0| \leq \delta^{\frac{2}{5}}$ for all q, thus using (2.8.12) and (2.8.53) we would deduce that

$$|Y^0 - W^0 s| \leq \delta + \delta^{\frac{2}{5}} t \tag{2.8.55}$$

for some $s > (t^0 - t^l)$, which is forbidden, for δ small, either by $\mathbf{z}_j \in \Omega_j$ (case $Y^0 \neq 0$) or by definition of $\mathbb{1}_0^\delta$ (case $Y^0 = 0$).

Eq. (2.8.53) implies $|Y^l \wedge \hat{W}^l| \leq \delta$, where $\hat{W}^l = \frac{W^l}{|W^l|}$. Using again (2.8.12) we have

$$\begin{aligned}
\delta &\geq \left| Y^0 \wedge \hat{W}^l - \sum_{q=0}^{l-1} (W^q \wedge \hat{W}^l)(t^q - t^{q+1}) \right| \\
&= \left| (Y^0 - W^0 t^0) \wedge \hat{W}^l - \sum_{q=1}^{l} [(W^q - W^{q-1}) \wedge \hat{W}^l)] t^q \right| \ .
\end{aligned} \tag{2.8.56}$$

Since the vectors involved in this relation do not depend on the times (Remark on page 49), we can exploit the integration in the variables t^q to estimate the set defined by this condition. However, a singularity will arise when the vector in the square brackets is small for all q. Let us focus first in this case, which is the most delicate. Assume that

$$\sum_{q=1}^{l} |(W^q - W^{q-1}) \wedge \hat{W}^l| \leq \delta^{\frac{3}{5}} \ . \tag{2.8.57}$$

Notice that if Eq. (2.8.57) is not satisfied, then $|(W^{q^*} - W^{q^*-1}) \wedge \hat{W}^l| > \delta^{\frac{3}{5}}/l$ for some q^*.

Again we may infer that condition (2.8.57) is not possible in the case $t^0 = t, Y^0 \neq 0$. Indeed, the above inequality trivially implies

$$|W^0 \wedge \hat{W}^l| \leq \delta^{\frac{3}{5}} \ , \tag{2.8.58}$$

therefore (2.8.56) gives

$$|Y^0 \wedge \hat{W}^l| \leq \delta + 2t\delta^{\frac{3}{5}} \ . \tag{2.8.59}$$

Putting together the two last equations we have

$$|Y^0 \wedge W^0| \leq C \left(|Y^0| + |W^0| \right) \delta^{\frac{3}{5}} \ , \tag{2.8.60}$$

which is excluded, for δ small, by $\mathbf{z}_j \in \Omega_j$.

Let us study (2.8.57) in the case $t^0 < t, Y^0 = 0$. By (2.8.54), there exists a $\overline{q} \in \{1, \cdots, l\}$ such that

$$U \equiv U^{\overline{q}} := W^{\overline{q}} - W^{\overline{q}-1} \tag{2.8.61}$$

has modulus

$$|U| > \frac{\delta^{\frac{2}{5}}}{l} \ . \tag{2.8.62}$$

But $|U \wedge \hat{W}^l| \leq \delta^{\frac{3}{5}}$ by (2.8.57). Then putting $\hat{U} = \frac{U}{|U|}$ we have

$$|\hat{U} \wedge \hat{W}^l| \leq n\delta^{\frac{1}{5}} \ . \tag{2.8.63}$$

On the other hand, by (2.8.58), either

$$|W^0| \leq \delta^{\frac{2}{5}} \ , \tag{2.8.64}$$

or $|\hat{W}^0 \wedge \hat{W}^l| \leq \delta^{\frac{1}{5}}$ which, together with (2.8.63) and the constraint $n \leq \overline{n} = O(|\log \varepsilon|)$, finally gives

$$|\hat{W}^0 \wedge \hat{U}| \leq C\delta^{\frac{1}{5}} |\log \varepsilon| \ . \tag{2.8.65}$$

We will use this formula to estimate the considered events, taking advantage from the fact that U depends only on the impact vector and relative velocity describing the interaction occurring at time $t^{\overline{q}}$ in the BBF (as already pointed out on page 59 and Figure 2.8), and that W^0 is the (incoming) relative velocity of the interaction at time t^0.

We shall summarize the discussion above as follows. Denote V_r and V_r' respectively the outgoing and incoming relative velocities of the collision at time t_r in the BBF. If $t^{\overline{q}} = t_r$, we use the notation $U_r = U^{\overline{q}}$. This is a function of ν_r, V_r only. We have

$$\mathbb{1}_0^\delta \mathbb{1}_{\mathcal{N}(\delta)} \leq \sum_{i,h} \sum_{l=1}^{n_{ih}} \sum_{q^*=1}^{l} \mathbb{1}_{\mathcal{N}_{ih}^{l,q^*}(\delta)} + \sum_{r=1}^{n} \mathbb{1}_{\{|V_r| \leq \delta^{2/5}\}} + \sum_{f=1}^{n} \sum_{f'=f+1}^{n} \mathbb{1}_{\{|\hat{V}_f' \wedge \hat{U}_{f'}| \leq C\delta^{1/5} |\log \varepsilon|\}} \tag{2.8.66}$$

where $1 \leq i < h \leq j+n$ and

$$\mathcal{N}_{ih}^{l,q^*}(\delta) = \Big\{ \mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n} \ \Big| \ \text{the virtual trajectories of } i \text{ and } h \text{ satisfy (2.8.56),} \\ \text{with } |(W^{q^*} - W^{q^*-1}) \wedge \hat{W}^l| > \delta^{\frac{3}{5}}/l \Big\} \ . \tag{2.8.67}$$

Once fixed all the variables but t^{q^*}, if Equation (2.8.56) is verified, then t^{q^*} belongs to an interval of length smaller than $\delta |(W^{q^*} - W^{q^*-1}) \wedge \hat{W}^l|^{-1}$. If we are in $\mathcal{N}_{ih}^{l,q^*}(\delta)$ this is bounded by $n\delta^{\frac{2}{5}}$, so that

$$\int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \mathbb{1}_{\mathcal{N}_{ih}^{l,q^*}(\delta)} e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \leq C^n \frac{t^{n-1}}{(n-1)!} \delta^{\frac{2}{5}} \ . \tag{2.8.68}$$

Changing variable $v_{j+r} \to V_r$ we easily find

$$\int d\Lambda(\mathbf{t}_n, \boldsymbol{\nu}_n, \mathbf{v}_{j,n}) \mathbb{1}_{\{|V_r| \leq \delta^{2/5}\}} e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \leq C^n \frac{t^n}{n!} \delta^{\frac{6}{5}} \ . \tag{2.8.69}$$

Furthermore, it is

$$\int d\Lambda(\mathbf{t}_n,\boldsymbol{\nu}_n,\mathbf{v}_{j,n})\mathbb{1}_{\{|\hat V'_f\wedge\hat U_{f'}|\le C\delta^{1/5}|\log\varepsilon|\}}e^{-\frac{\beta}{2}\sum_{i=1}^{j+n}v_i^2}\le C^n\frac{t^n}{n!}\delta^{\frac25}|\log\varepsilon|^{\frac72}\,. \tag{2.8.70}$$

To prove this last inequality, it is convenient first to introduce a further restriction to the set $\{\frac{\beta}{2}|V_{f'}|^2<4|\log\varepsilon|\}$ where $V_{f'}=(v_{j+f'}-\eta_{k_{f'}}(t^+_{f'}))$. If the opposite inequality holds, then either $|v_{j+f'}|$ or $|\eta_{k_{f'}}(t_{f'}))|$ cannot be smaller than $\sqrt{2/\beta|\log\varepsilon|}$, hence the total energy must be larger than $1/\beta|\log\varepsilon|$. Therefore, using the energy–cutoff $\mathbb{1}^\varepsilon_1$ defined in (2.8.22), the error produced is bounded by

$$\begin{aligned}&\int d\Lambda(\mathbf{t}_n,\boldsymbol{\nu}_n,\mathbf{v}_{j,n})\,(1-\mathbb{1}^\varepsilon_1)\,e^{-\frac{\beta}{2}\sum_{i=1}^{j+n}v_i^2}\\&\le e^{-\frac12|\log\varepsilon|}\int d\Lambda(\mathbf{t}_n,\boldsymbol{\nu}_n,\mathbf{v}_{j,n})e^{-\frac{\beta}{4}\sum_{i=1}^{j+n}v_i^2}\le C^n\frac{t^n}{n!}\varepsilon^{\frac12}\,,\end{aligned} \tag{2.8.71}$$

which is in turn certainly smaller than $(Ct)^n/n!\delta^{\frac25}$, being δ given by (2.8.25). We are left with

$$\int d\Lambda(\mathbf{t}_n,\boldsymbol{\nu}_n,\mathbf{v}_{j,n})\mathbb{1}_{\{|\hat V'_f\wedge\hat U_{f'}|\le C\delta^{1/5}|\log\varepsilon|\}}\mathbb{1}_{\{\frac{\beta}{2}|V_{f'}|^2<4|\log\varepsilon|\}}e^{-\frac{\beta}{2}\sum_{i=1}^{j+n}v_i^2}\,. \tag{2.8.72}$$

We change the integration variables according to (2.8.18)–(2.8.19). Note that

$$e^{-\frac{\beta}{2}v^2_{j+f}}e^{-\frac{\beta}{2}v^2_{j+f'}}=e^{-\frac{\beta}{2}\left(V'_f-2\omega_f(\omega_f\cdot V'_f)+\eta_{k_f}(t^+_f)\right)^2}e^{-\frac{\beta}{2}\left(V_{f'}+\eta_{k_{f'}}(t^+_{f'})\right)^2}\,, \tag{2.8.73}$$

where $\omega_f=\omega(\nu'_f,V'_f)$ is the scattering vector at the collision. Since $V_{f'}$ varies in a compact set, we bound the second exponential simply by 1, while the first exponential is estimated by $e^{-\frac{\beta}{2}\left(|V'_f|-|\eta_{k_f}(t^+_f)|\right)^2}$, where $\eta_{k_f}(t^+_f)$ depends only on the variables with index strictly smaller than f. Performing first the integrations in $dV_{f'}dV'_f$ we find

$$\begin{aligned}&\int dV_{f'}\mathbb{1}_{\{\frac{\beta}{2}|V_{f'}|^2<4|\log\varepsilon|\}}\int dV'_f\mathbb{1}_{\{|\hat V'_f\wedge\hat U_{f'}|\le C\delta^{1/5}|\log\varepsilon|\}}e^{-\frac{\beta}{2}\left(|V'_f|-|\eta_{k_f}(t^+_f)|\right)^2}\\&\le\int dV_{f'}\mathbb{1}_{\{\frac{\beta}{2}|V_{f'}|^2<4|\log\varepsilon|\}}\left(C\delta^{\frac25}|\log\varepsilon|^2\right)\int d|V'_f||V'_f|^2e^{-\frac{\beta}{2}\left(|V'_f|-|\eta_{k_f}(t^+_f)|\right)^2}\\&\le\left(C|\log\varepsilon|^{\frac32}\right)\left(C\delta^{\frac25}|\log\varepsilon|^2\right)\left(C(1+|\eta_{k_f}(t^+_f)|^2)\right)\\&\le C\left(1+\sum_{i=1}^{j+f-1}v_i^2\right)\delta^{\frac25}|\log\varepsilon|^{\frac72}\,.\end{aligned} \tag{2.8.74}$$

Integrating in the remaining variables we readily get Equation (2.8.70).

Collecting all the estimates, Lemma 6 is proved. □

Substituting Eq. (2.8.47) in Eq. (2.8.46), performing the sums and using (2.8.43), we conclude that

$$|\mathcal{E}_5|\le C^j\varepsilon^{-\lambda}\delta^{\frac25}|\log\varepsilon|^{\frac72}\le\varepsilon^{\frac25-\frac25\mu-\lambda}|\log\varepsilon|^{\frac{43}{10}} \tag{2.8.75}$$

for t small enough.

We turn now to the estimates of the errors coming from the truncations defined in (2.8.22). Proceeding as above we have

$$
\begin{aligned}
|\mathcal{E}_6| \leq & \sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} \Big[\int d\Lambda \ \mathbb{1}_{\{(\prod B^\varepsilon)\leq \varepsilon^{-\lambda}\}} \left(\prod B^\varepsilon\right) \left(1-\mathbb{1}_{\mathcal{N}(\delta)}\right) \left(1-\mathbb{1}_1^\varepsilon \mathbb{1}_2^\varepsilon\right) f_{0,j+n}^N \\
& + \int d\Lambda \ \mathbb{1}_{\{(\prod B)\leq \varepsilon^{-\lambda}\}} \left(\prod B\right) \left(1-\mathbb{1}_{\mathcal{N}(\delta)}\right) \left(1-\mathbb{1}_1^\varepsilon \mathbb{1}_2^\varepsilon\right) f_{0,j+n} \Big] \\
\leq & \ \varepsilon^{-\lambda} \sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} C^{j+n} \int d\Lambda \ \left(1-\mathbb{1}_1^\varepsilon \mathbb{1}_2^\varepsilon\right) e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \\
\leq & \ C^j \varepsilon^{\frac{1}{2}-\lambda} + \varepsilon^{-\lambda} \sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} C^{j+n} \int d\Lambda \ \left(1-\mathbb{1}_2^\varepsilon\right) e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \ ,
\end{aligned}
\tag{2.8.76}
$$

where in the last step we used (2.8.71). Moreover

$$
\int d\Lambda \left(1-\mathbb{1}_2^\varepsilon\right) e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \leq \sum_{r=1}^{n} \int d\Lambda \ \mathbb{1}_{\{|(v_{j+r}-\eta_{k_r}^\varepsilon(t_r))\wedge \nu_r|\leq \varepsilon^\mu\}} e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \ . \tag{2.8.77}
$$

We perform first the integrations in $dv_{j+r}d\nu_r$. Setting $V_r = (v_{j+r} - \eta_{k_r}^\varepsilon(t_r))$, α the angle between V_r and ν_r and using the parametrization $\nu_r \to (\rho, \phi)$ where $\rho = \sin\alpha$ and $\phi \in [0, 2\pi)$, we have

$$
\begin{aligned}
& \int dv_{j+r} d\nu_r \mathbb{1}_{\{|(v_{j+r}-\eta_{k_r}^\varepsilon(t_r))\wedge \nu_r|\leq \varepsilon^\mu\}} e^{-\frac{\beta}{2}v_{j+r}^2} \\
& = \int dv_{j+r} (2\pi) 2 \int_0^1 d\rho \frac{\rho}{\sqrt{1-\rho^2}} \mathbb{1}_{\{|(v_{j+r}-\eta_{k_r}^\varepsilon(t_r))|\rho\leq \varepsilon^\mu\}} e^{-\frac{\beta}{2}v_{j+r}^2} \\
& \leq \int dv_{j+r} (2\pi) 2 \int_0^{\varepsilon^\mu} d\rho \frac{\rho}{\sqrt{1-\rho^2}} e^{-\frac{\beta}{2}v_{j+r}^2} + (2\pi) 2 \int_{\varepsilon^\mu}^1 d\rho \frac{\rho}{\sqrt{1-\rho^2}} \int_0^{\frac{\varepsilon^\mu}{\rho}} d|V_r| 4\pi |V_r|^2 \\
& \leq C\varepsilon^{2\mu} \ .
\end{aligned}
\tag{2.8.78}
$$

The integrals in the other variables give $C^{n-1}t^n/n!$, therefore performing the sums as above we obtain

$$
|\mathcal{E}_6| \leq C^j \left(\varepsilon^{\frac{1}{2}-\lambda} + \varepsilon^{2\mu-\lambda}\right) \ . \tag{2.8.79}
$$

Now we shall estimate what is left of the differences in (2.8.34). This gives two errors: one is due to the convergence of the initial data (formula (2.5.7) in Hypothesis 5) and the other is due to the convergence of the IBF to the BBF (Lemma 5). The first is

$$
\begin{aligned}
|\mathcal{E}_7| \leq \sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} \int d\Lambda \ \mathbb{1}_{\{(\prod B^\varepsilon)\leq \varepsilon^{-\lambda}\}} \left(\prod B^\varepsilon\right) \left(1-\mathbb{1}_{\mathcal{N}(\delta)}\right) \mathbb{1}_1^\varepsilon \mathbb{1}_2^\varepsilon \\
\cdot \left| f_{0,j+n}^N(\boldsymbol{\zeta}^\varepsilon(0)) - f_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)) \right| \ .
\end{aligned}
\tag{2.8.80}
$$

Since we integrate outside $\mathcal{N}(\delta)$, the BBF satisfies $\boldsymbol{\zeta}(0) \in \mathcal{M}_j(\delta)$. But $\delta = \varepsilon^{1-\mu}(\log \varepsilon)^2$. Thus, applying Lemma 5, the IBF must satisfy $\boldsymbol{\zeta}^\varepsilon(0) \in \mathcal{M}_j(\varepsilon)$ for ε small enough. Hence Hypothesis 5 together with conservation of energy lead to

$$\begin{aligned} |\mathcal{E}_7| &\leq \varepsilon^{1-\lambda} \sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} C^{j+n} \int d\Lambda\ e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \\ &\leq C^j \varepsilon^{1-\lambda}\ , \end{aligned} \tag{2.8.81}$$

having performed the sums in the usual way.

Finally, the last error is

$$\begin{aligned} |\mathcal{E}_8| \leq \sum_{n=0}^{\overline{n}} \sum_{\Gamma(j,n)} \sum_{\boldsymbol{\sigma}_n} \int d\Lambda\ \mathbb{1}_{\{(\prod B^\varepsilon) \leq \varepsilon^{-\lambda}\}} \left(\prod B^\varepsilon \right) \left(1 - \mathbb{1}_{\mathcal{N}(\delta)}\right) \mathbb{1}_1^\varepsilon \mathbb{1}_2^\varepsilon \\ \cdot \left| f_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)) - f_{0,j+n}(\boldsymbol{\zeta}(0)) \right| . \end{aligned} \tag{2.8.82}$$

Lemma 5, the regularity assumption (2.5.8) in Hypothesis 5 and conservation of energy at collisions imply that we can bound the modulus in the last line by

$$C^{j+n} e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} \left(\sum_{k=1}^{j+n} |\xi_k^\varepsilon(0) - \xi_k(0)|^2 \right)^{\frac{1}{2}} \leq C^{j+n} e^{-\frac{\beta}{2}\sum_{i=1}^{j+n} v_i^2} (j+n)^{\frac{1}{2}} D \varepsilon^{1-\mu} |\log \varepsilon|^{\frac{3}{2}}\ , \tag{2.8.83}$$

for ε sufficiently small. Therefore, proceeding as above we have

$$|\mathcal{E}_8| \leq C^j \varepsilon^{1-\mu-\lambda} |\log \varepsilon|^{\frac{3}{2}}\ . \tag{2.8.84}$$

Putting together all the errors $\mathcal{E}_1, \cdots, \mathcal{E}_8$ and optimizing on μ and λ we conclude that

$$|f_j^N(\mathbf{z}_j, t) - f_j(\mathbf{z}_j, t)| \leq C^j \varepsilon^\gamma \qquad \text{for any} \qquad \gamma < \frac{1}{6}\ . \tag{2.8.85}$$

□

2.9 Stable potentials

In this section we show how the techniques used in proving Theorem 1 can be extended to treat a fairly larger class of potentials, including those with an attractive part.

The potentials Φ considered in the present section satisfy the following conditions.

Hypothesis 1′ *$\Phi = \Phi(q), q \in \mathbb{R}^3$ is radial, with support $|q| < 1$. We further assume*
1) either $\Phi \in C^2(\mathbb{R}^3)$, or $\Phi \in C^2(\mathbb{R}^3 \setminus \{0\})$ and $\Phi(q) \to +\infty$ as $q \to 0$;
2) Φ is stable.

In what follows we will use the usual notational inconsistency $\Phi(r) = \Phi|_{|q|=r}$.

We remind (see e.g. [14]) that an interaction is stable if it fulfills the following condition:

$$U(q_1, \cdots, q_j) = \sum_{i<h} \Phi(|q_i - q_h|) \geq -Bj \tag{2.9.1}$$

for some constant $B > 0$. In particular, Φ is positive (possibly diverging) at the origin. We also remark that condition (2.9.1) ensures the existence of the Partition Function and hence the existence of an equilibrium measure. In our context the stability will be used to guarantee that Hypothesis 3 implies the bound $f_{0,j}^N \leq c^j e^{-\frac{\beta}{2}\sum_i v_i^2}$ (where $c = e^{\alpha+\beta B}$), which is crucial in our proof.

The potentials Φ we are considering include a sort of truncated Lennard-Jones potential (see Fig. 2.9 below).

Figure 2.9: A cutoffed Lennard–Jones potential.

We note that the proof presented in Section 2.8.2 depends on Φ only through the scattering time τ_*. Therefore, in trying to extend it to the present situation, the crucial point will be the control of τ_*, in absence of Lemma 1 of Sec. 2.3.2 which is not valid anymore.

According to Section 2.3.2, we reduce the two–body particle system to the central motion of a single particle, with mass $\frac{1}{2}$ and velocity V (the relative velocity of the two interacting particles).

We recall the formula yielding the interaction time:

$$\tau_* = \sqrt{2} \int_{r_*}^{1} dr \frac{1}{\left(\frac{V^2}{2} - \frac{L^2}{2r^2} - 2\Phi(r)\right)^{1/2}} \,, \tag{2.9.2}$$

where $|V| > 0$ is the modulus of the relative velocity before the collision, ρ is the impact parameter, $L = |\rho V| \in [0, |V|]$ is the modulus of the angular momentum and r_* is the infimum of the distance from the origin during the scattering process. r_* is given by

$$r_* = \max\left\{x \in [0,1) \ \text{ s.t. } \frac{V^2}{2} = \frac{L^2}{2x^2} + 2\Phi(x)\right\}. \tag{2.9.3}$$

Before establishing the following lemma in which we control the scattering time, we discuss the new difficulties we face in presence of an attractive part. Consider for instance the potentials described in Fig. 2.9, with a single negative well. The effective potential

$$\Phi(r) + \frac{L^2}{4r^2} - \frac{L^2}{4} \tag{2.9.4}$$

may have two critical points, r_m and r_M (minimum and maximum respectively), when L is sufficiently small; see Figure 2.10. Fixing a value of L for which such critical points do exist,

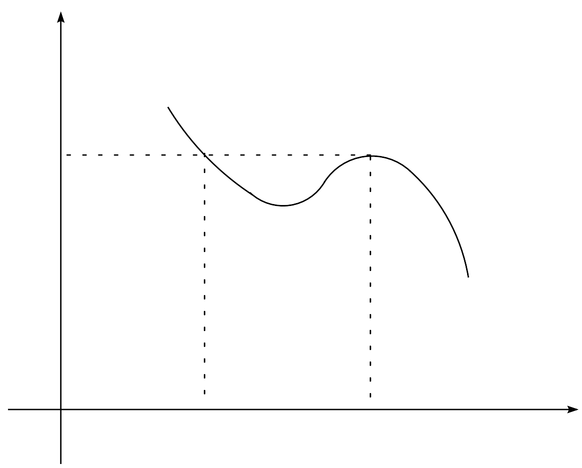

Figure 2.10: When the (real) interaction has attractive parts, the effective potential (drawn in figure) can have local maxima for a given value of L.

there are values of V for which

$$\frac{V^2}{2} \approx 2\Phi(r_M) + \frac{L^2}{2r_M^2}. \tag{2.9.5}$$

In this case the trajectory is close to an unstable periodic orbit and τ_* is very large. The two particles turn around each other many times and remain trapped for a long time. Clearly such situations are pathological and must be excluded in order to have a kinetic picture. Actually the following lemma says that the set of such pathological events has a small measure, although we do not give explicit estimates.

Lemma 7. *Given $\eta \in (0,1)$ and $K \in \mathbb{R}^+$, there exists a set $\mathcal{B}(\eta)$ of pairs (ν, V) such that*

$$\int_{S^2} d\nu \int_{\{\nu\cdot V\leq 0\}} dV|\nu\cdot V|\mathbb{1}_{\mathcal{B}(\eta)} \to 0 \tag{2.9.6}$$

as $\eta \to 0$ and such that, for $(\nu, V) \notin \mathcal{B}(\eta)$ and $|V| < K$,

$$\tau_*(\nu, V) < \varphi(\eta, K) \ , \tag{2.9.7}$$

where $\varphi(\eta, K)$ is a positive function which may possibly diverge as $\eta \to 0$ or $K \to +\infty$.

Proof. We can easily find the set of pairs (L^2, V^2) for which τ_* diverges. Such a set is included in the set for which there exist local maxima of the effective potential (2.9.4). The critical points satisfy

$$\Phi'(y) - \frac{L^2}{2y^3} = 0 \ . \tag{2.9.8}$$

Therefore the pairs (L^2, V^2) corresponding to a divergence of the scattering time τ_* must satisfy (2.9.8) and

$$\frac{V^2}{2} = 2\Phi(y) + \frac{L^2}{2y^2} \ . \tag{2.9.9}$$

This last condition is due to Eq. (2.9.3), while (2.9.8) ensures that the orbit reaches $r_* = y$ in an infinite time.

Consider now the curve $\mathcal{C}$ in the plane, $y \in (0,1) \to (X, Y)$, whose parametric equations are

$$\begin{cases} X = 2\Phi'(y)y^3 \\ Y = 4\Phi(y) + 2\Phi'(y)y \end{cases} . \tag{2.9.10}$$

Then the set of singular values of (L^2, V^2) lies inside the restriction $\tilde{\mathcal{C}}$ of this curve (X, Y) to the"physical"subset

$$\Big\{(X, Y) \ \text{ s.t. } \ Y > 0, \ 0 \le X \le Y\Big\} \ . \tag{2.9.11}$$

Clearly when Φ is bounded the curve $\mathcal{C}$ is extended by continuity to $y = 0$ (for which $L^2 = X(0) = 0, V^2 = Y(0) = 4\Phi(0)$ are indeed singular points of τ_*). Note that, when Φ is unbounded, the parameter y spanning $\tilde{\mathcal{C}}$ is bounded away from zero, since it cannot be smaller than $r_0 := \min\{x \in (0,1] \ \text{ s.t. } \Phi'(x) \ge 0\}$.

Denoting by $B((X, Y); \eta)$ the disk of center (X, Y) and radius η, we introduce the tube

$$\mathcal{T}(\eta) = \bigcup_{(X,Y) \in \mathcal{C}} B((X, Y); \eta) \tag{2.9.12}$$

and its restriction to the physical region

$$\tilde{\mathcal{T}}(\eta) = \bigcup_{(X,Y) \in \tilde{\mathcal{C}}} B((X, Y); \eta) \ . \tag{2.9.13}$$

Now observe that, due to the smoothness of Φ, the set $\tilde{\mathcal{C}}$ has finite length so that

$$|\tilde{\mathcal{T}}(\eta)| \le C\eta \ , \tag{2.9.14}$$

where $|A|$ denotes the Lebesgue measure of the set A.

Consider the set

$$\tilde{\mathcal{T}}(\eta) \cup B((0,0);\eta) \cup \{|V| \geq K\} \,. \tag{2.9.15}$$

Its complement $\mathcal{G}(\eta, K)$ in the physical region (2.9.11) is relatively compact. Therefore, by continuity of $\tau_*(L^2, V^2)$ in the set (2.9.11) deprived of $\tilde{\mathcal{C}}$ (and hence in the closure of $\mathcal{G}$), we have

$$\tau_* < \varphi(\eta, K) \tag{2.9.16}$$

in $\mathcal{G}(\eta, K)$, for a suitable positive function φ, possibly diverging as $\eta \to 0$ or $K \to +\infty$.

To prove the required continuity of τ_*, we observe first that $r_* = r_*(L^2, V^2)$ is continuous outside $\tilde{\mathcal{C}}$. Fix a point $(L_0^2, V_0^2) \notin \tilde{\mathcal{C}}$ in the set (2.9.11). Then, for any $\gamma \in \left(0, \frac{1-r_*(L_0^2,V_0^2)}{2}\right)$, the integral

$$\sqrt{2}\int_{r_*+\gamma}^{1} dr \frac{1}{\left(\frac{V^2}{2} - \frac{L^2}{2r^2} - 2\Phi(r)\right)^{1/2}} \tag{2.9.17}$$

is continuous in (L_0^2, V_0^2), being the integrand bounded. On the other hand,

$$\sqrt{2}\int_{r_*}^{r_*+\gamma} dr \frac{1}{\left(\frac{V^2}{2} - \frac{L^2}{2r^2} - 2\Phi(r)\right)^{1/2}} \longrightarrow 0 \tag{2.9.18}$$

as $\gamma \to 0$, uniformly for $(L^2, V^2) \in B((L_0^2, V_0^2); \delta)$, if δ is small enough. This can be seen by an argument as the one in Lemma 1, namely using the estimate (2.3.21) replacing the integration interval $(r_*, 1)$ with $(r_*, r_* + \gamma)$.

To conclude the proof of the lemma, we introduce the set

$$\mathcal{B}(\eta) = \left\{(\nu, V) \in S^2 \times \mathbb{R}^3 \;\middle|\; (L^2, V^2) \in \tilde{\mathcal{T}}(\eta) \cup B((0,0);\eta)\right\} \tag{2.9.19}$$

where $L = |\nu \wedge V|$. Setting $\cos\alpha = \nu \cdot \hat{V}$, $\hat{V} = V/|V|$ and noticing that $L = |V \sin\alpha|$, the left hand side of (2.9.6) is bounded by

$$\begin{aligned}\int_{S^2} d\nu \int_{S^2} d\hat{V} \int_0^\infty d|V||V|^3 |\cos\alpha| \mathbb{1}_{\mathcal{B}(\eta)} &= 8\pi^2 \int_0^\infty d|V||V|^3 \int_0^\pi d\alpha \sin\alpha |\cos\alpha| \mathbb{1}_{\mathcal{B}(\eta)} \\ &\leq 4\pi^2 \int_0^\infty dV^2 \int_0^{V^2} dL^2 \mathbb{1}_{\mathcal{B}(\eta)} \leq C\eta \end{aligned} \tag{2.9.20}$$

for η sufficiently small. $\square$

We are now in a position to establish and prove the main result of the present section.

Theorem 1′ *(Improved) Under the Hypotheses* 1′ *and 2–4 of Section 2.5, there exists* $t_0 > 0$ *such that, for* $0 < t < t_0$ *and* $j > 0$,

$$\lim_{\substack{\varepsilon \to 0 \\ N\varepsilon^2 = 1}} f_j^N(t) = f_j(t) \tag{2.9.21}$$

uniformly on compact sets in Ω_j.

Proof. We just mention where the previous proof of Theorem 1 requires modifications and how to do them.

The proof consists of two parts, namely the short time estimate and the term by term convergence.

As regards the short time bound in Section 2.6, by virtue of the stability property, it is natural to modify the definition of the Hamiltonian by setting

$$H_B(\mathbf{z}_j) = H(\mathbf{z}_j) + jB \geq \frac{1}{2}\sum_{i=1}^{j} v_i^2 \geq 0 \; . \tag{2.9.22}$$

Consequently we introduce the norms (2.6.1) replacing H by H_B. Next we deduce estimate (2.6.3) by using (2.9.22) and the fact that H_B satisfies the inequality (2.6.5), i.e.

$$H_B(\mathbf{z}_{j+1+m}) = H_B(\mathbf{z}_j) + H_B(\mathbf{z}_{j,1+m}) \geq H_B(\mathbf{z}_j) + \frac{1}{2}\sum_{i=j+1}^{j+1+m} v_i^2 \; . \tag{2.9.23}$$

We use the stability of the potential only in this part of the proof.

Now we pass to analyze the term by term convergence. Everything is going on as in Section 7.2, with the only difference that we replace

$$\mathbb{1}_2^\varepsilon = \prod_{r=1}^{n} \mathbb{1}_{\{|(v_{j+r}-\eta^\varepsilon_{k_r}(t_r))\wedge \nu_r|>\varepsilon^\mu\}} \tag{2.9.24}$$

by

$$\mathbb{1}_2^\varepsilon = \prod_{r=1}^{n} \mathbb{1}_{\{(\nu_r, v_{j+r}-\eta^\varepsilon_{k_r}(t_r))\notin \mathcal{B}(\eta), |v_{j+r}-\eta^\varepsilon_{k_r}(t_r)|<K\}} \; . \tag{2.9.25}$$

According to the form of the function φ, we can choose $\eta = \eta(\varepsilon)$ and $K = K(\varepsilon)$ such that $\eta \to 0$ and $K \to \infty$ with ε and the scattering time of the collisions associated to the nodes of the tree (in macroscopic variables) is bounded by $\varepsilon\varphi(\eta, K) \leq A\varepsilon^{1-\mu}$ whenever $\mathbb{1}_2^\varepsilon = 1$ (by virtue of Lemma 7). Therefore Lemma 5 (hence Eq. (2.8.24)) is still valid and the proof is completed by observing that, by Lemma 7, $B(\eta)$ is a set of vanishing measure (so Eq. (2.8.23) holds). □

2.10 Concluding remarks

We conclude by discussing some additional remarks.

1. The potentials we have considered are fairly general, but the basic hypothesis is the short-range assumption.

From the very beginning of the Kinetic Theory, Boltzmann himself (see [1]), following Maxwell [12, 13], considered only inverse power law potentials, besides the hard–sphere system, originally investigated in deriving his famous equation. This is probably due to the good scaling properties of such potentials. Moreover the differential cross–section is well defined, even though the total cross-section is diverging because of the long range of the interaction. On the other hand, it is not clear whether the Boltzmann equation associated to these potentials can indeed be derived under the low–density scaling.

A simpler problem would be to consider a sequence of potentials with range gently diverging with N. This problem eludes the present techniques so that we consider it as an interesting, open problem.

For an analysis concerning the much easier problem of the validity of the linear Boltzmann equation for long range potentials, see reference [4].

2. In the present paper we give an explicit estimate of the error in case of a completely repulsive potential (Theorem 2), while, for stable potentials, we only show the convergence. It would be interesting to develop a constructive proof of convergence also in this last case. This would require a more precise estimate of the scattering time to improve Lemma 7.

3. The present validity results, as the ones in the previous literature, are formulated in a canonical context, namely, for any $\varepsilon > 0$, the number of particles N is authomatically fixed. An equivalent formalism is the grand–canonical one. Here the number of particles is random but the density is fixed.

More precisely consider, for a given ε, the phase space of the system as

$$\mathcal{M} = \bigcup_{N\geq 0} \mathcal{M}_N \tag{2.10.1}$$

where $\mathcal{M}_N$ is the N–particle phase space (see (2.3.8)). For $z \in \mathcal{M}$ we define the dynamical flow by solving the Newton equations in each $\mathcal{M}_N$. Similarly we define a symmetric probability measure W^ε on $\mathcal{M}$ by means of a sequence of symmetric probability measures W^N in each $\mathcal{M}_N$:

$$W^\varepsilon|_{\mathcal{M}_N} = e^{-\mu_\varepsilon} \frac{\mu_\varepsilon^N}{N!} W^N. \tag{2.10.2}$$

The sequence $\{g_j^\varepsilon\}_{j=1}^\infty$ is defined by

$$g_j^\varepsilon = \sum_{N\geq j} e^{-\mu_\varepsilon} \frac{\mu_\varepsilon^N}{N!} g_j^N \tag{2.10.3}$$

where g_j^N are the marginals of W^N. Therefore $g_j^\varepsilon(\mathbf{z}_j)$ are the probability densities of finding the first j particles in $\mathbf{z}_j$. Their normalizations are

$$\sum_{N\geq j} e^{-\mu_\varepsilon} \frac{\mu_\varepsilon^N}{N!} \tag{2.10.4}$$

which is the probability of finding more than j particles. Then one defines the reduced marginals accordingly and it is easy to derive the equivalent of the Grad hierarchy for them.

Note now that the average number of particles is $\langle N \rangle = \mu_\varepsilon$. Therefore the low–density limit will correspond to $\mu_\varepsilon \to \infty$ and $\varepsilon^2 \mu_\varepsilon \to l^{-1} > 0$.

It is not difficult to realize that the validity result of this paper can be formulated and proven also in this context.

4. We have considered the particle system in the whole space. If we want the system to be confined in a bounded box, we have to specify the boundary conditions. Assuming specular reflections, there are additional difficulties which we have to overcome. First the dynamical flow is only almost everywhere defined (see [11]), but this (as for the hard–sphere systems) does not create real difficulties. However the analysis of the recollisions requires some extra geometrical arguments.

2.11 Appendix (on the cross–section for the Boltzmann equation)

In this appendix we give sufficient conditions on the interaction for having a single–valued differential cross–section (and we show some counterexample). We also study the boundedness properties of the cross–section. The issue is relevant both to motivate our strategy and to know whether B is a well behaved single–valued function in the usual form of the Boltzmann equation, Eq. (2.2.3).

The assumptions on Φ are those established in Hypothesis 1$'$, but possibly allowing a discontinuity of the first derivative at $|q| = 1$.

Consider the planar scattering process of a particle of unit mass. We use the notations of Section 2.3.2 and of Figure 2.1. In particular, we denote by ρ the impact parameter (by symmetry we may focus on $0 \leq \rho \leq 1$) while the scattering angle is $\chi = \pi - 2\Theta$ and the energy in the laboratory $V^2/2 > 0$.

The differential cross–section is defined through the map $\rho = \rho(\Theta, |V|)$, by

$$\sigma_\Phi = \frac{\rho}{2|\sin(2\Theta)|}\left|\frac{d\rho}{d\Theta}\right| . \tag{2.11.1}$$

Therefore we need to analyze invertibility of the map $\Theta(\rho)$.

The classical formula for Θ is

$$\Theta(\rho) = \arcsin\rho + \rho\int_{r_*}^{1} dr \frac{1}{r^2\sqrt{1 - \frac{2\Phi(r)}{V^2} - \frac{\rho^2}{r^2}}} , \tag{2.11.2}$$

where r_* is the minimum distance of the central motion from the origin, satisfying

$$1 - \frac{2\Phi(r_*)}{V^2} - \frac{\rho^2}{r_*^2} = 0 . \tag{2.11.3}$$

For purely repulsive potentials with a singularity at the origin, the limiting values are $\Theta(0) = 0$ and $\Theta(1) = \pi/2$. In general, it is $\Theta = n\pi + \Theta'$ for some $\Theta' \in (0, \pi)$, where n is the total number of counterclockwise turns that the trajectory makes around the origin (see Section 2.9).

While the first term in the right hand side of (2.11.2) is an increasing function of ρ, the second term is clearly non monotonic (in fact it goes smoothly from 0 to 0 when $\rho \to 0$ or $\rho \to 1$ and hence $r_* \to 1$). Following [4], we set $y = \rho/r$ and perform the change of variables

$$\frac{2\Phi(\frac{\rho}{y})}{V^2} + y^2 = \sin^2\varphi \; , \tag{2.11.4}$$

to get

$$\Theta(\rho) = \arcsin\rho + \int_{\arcsin\rho}^{\pi/2} d\varphi \frac{\sin\varphi}{y - \frac{\rho\Phi'(\frac{\rho}{y})}{V^2 y^2}} \; . \tag{2.11.5}$$

The advantage of this formula is that the integrand is not singular in the integration region and we can easily compute the derivative with respect to ρ.

A straightforward calculation leads to

$$\begin{aligned} \frac{d\Theta}{d\rho} &= \frac{1}{\sqrt{1-\rho^2}}\left(1 - \frac{1}{1 - \frac{\Phi'(1^-)}{V^2\rho^2}}\right) \\ &+ \int_{\arcsin\rho}^{\pi/2} d\varphi \frac{\sin\varphi}{\left(y - \frac{\rho}{V^2y^2}\Phi'(\frac{\rho}{y})\right)^3}\left[\frac{\rho}{V^2y^2}\Phi''\left(\frac{\rho}{y}\right) + \frac{2}{V^2y}\Phi'\left(\frac{\rho}{y}\right) + \frac{\rho}{V^4y^4}\left(\Phi'\left(\frac{\rho}{y}\right)\right)^2\right] \end{aligned} \tag{2.11.6}$$

for $0 < \rho < 1$, where $\Phi'(1^-)$ indicates the limit of the derivative as $|q| \to 1$ from below.

In formula (2.11.6) we are also considering the case in which Φ has a discontinuity of the first derivative in $|q| = 1$ as it is the case of the inverse power law potential restricted to the unitary interval treated in [4]. However, for the case of smooth potentials as the ones considered in the present paper, the first term in the right hand side of Eq. (2.11.6) is absent.

The following considerations can be deduced from Eq. (2.11.6).

1) The ratio $\rho/y \to g(\varphi)$ as $\rho \to 0$, where g is a positive function of φ which form depends on Φ and V^2. Then the extremal values of our derivative are:

$$\begin{aligned} &\frac{d\Theta}{d\rho} \underset{\rho\to 0}{\longrightarrow} (1 - \delta_{\Phi'(1^-),0}) + \int_0^{\pi/2} d\varphi \frac{g(\varphi)\sin\varphi}{\left(\frac{g(\varphi)}{V^2}|\Phi'(g(\varphi))|\right)^3}\frac{(\Phi'(g(\varphi)))^2}{V^4} \in (0, +\infty] \; ; \\ &\frac{d\Theta}{d\rho} \underset{\rho\to 1}{\longrightarrow} \begin{cases} +\infty, & \Phi'(1^-) \neq 0 \\ 0, & \Phi'(1) = 0 \end{cases} \; . \end{aligned} \tag{2.11.7}$$

2) The monotonicity property $\frac{d\Theta}{d\rho} > 0$ translates in a quite complicated condition on the potential Φ. A convenient sufficient condition is given by the following assertion: *In the considered class of potentials, if for all q with $|q| \in (0,1)$*

$$|q|\Phi''(|q|) + 2\Phi'(|q|) \geq 0 \; , \tag{2.11.8}$$

then $\frac{d\Theta}{d\rho} > 0$ for all $\rho \in (0,1), V^2 > 0$. This condition is derived also in [15].

Condition (2.11.8) can be easily checked for a large subset of potentials. For instance any potential of the form $\Phi(q) = \left(\frac{1}{|q|^k} - 1\right)\delta_{|q|<1}, k \geq 1$, satisfies the condition, hence has strictly monotonic map. Cases which are smooth in $|q| = 1$ can be constructed from the previous by using a smooth junction. For instance[3]

$$\Phi(q) = \begin{cases} e^{-\frac{1}{\delta}}\left(\frac{(1-\delta)^{k+1}}{\delta^2 k}\frac{1}{|q|^k} + 1 - \frac{1-\delta}{\delta^2 k}\right) & 0 < |q| < 1-\delta \\ e^{-\frac{1}{1-|q|}} & 1-\delta \leq |q| < 1\ , \\ 0 & |q| \geq 1 \end{cases} \tag{2.11.9}$$

where $k \geq 1$ and $0 < \delta < 1/3$ (see Fig. 2.11).

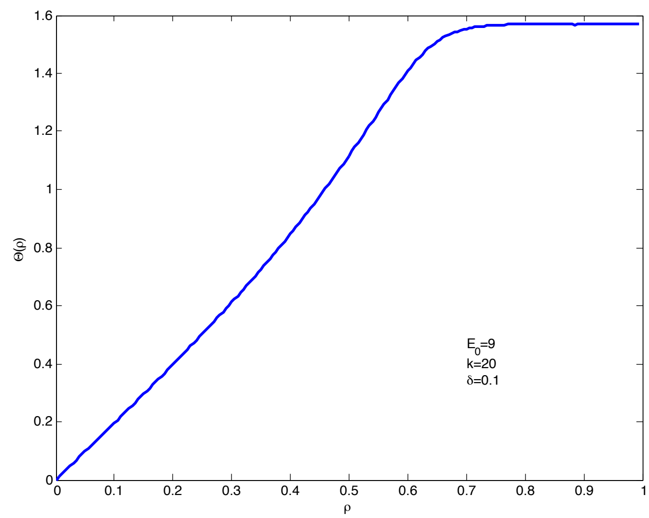

Figure 2.11: Map $\Theta(\rho)$ for the potential given by Eq. (2.11.9), with $\delta = 0.1, k = 20$ and $E_0 = V^2/2 = 9$.

3) The monotonicity property $\frac{d\Theta}{d\rho} > 0$ is in general not true when condition (2.11.8) is violated. A first example is any smooth, positive, decreasing and bounded potential (for which $\Theta(0) = \Theta(1) = 0$, that implies the existence of at least two monotonicity branches).

We give two different examples of potentials singular at the origin.

- Formula (2.11.6) indicates that the sign of the second derivative of Φ is relevant when we ask about monotonicity of the map. In fact, examples of non monotonic maps can be constructed when Φ'' is not always positive, for instance by taking Φ very close to the characteristic function of $|q| < 1$. If we consider the function

$$\Phi(q) = -\varepsilon \tan\left(\left(\arctan\frac{1}{\varepsilon} + \frac{\pi}{2}\right)|q| - \frac{\pi}{2}\right) + 1\ , \tag{2.11.10}$$

[3] Observe that this is a function $C^1(\mathbb{R}^d)$ with a jump in the second derivative for $|q| = 1-\delta$. The parameters k and δ can be arranged in order to eliminate this discontinuity (e.g. $\delta = 1/10, k = 71$). Nevertheless, all our discussions are still valid if in the initial assumptions on the potential we require that Φ'' is just piecewise continuous and bounded outside any ball centered in the origin.

numerical simulations show that the map $\Theta(\rho)$ is non monotonic for $\varepsilon << 1$ as shown in Fig. 2.12.

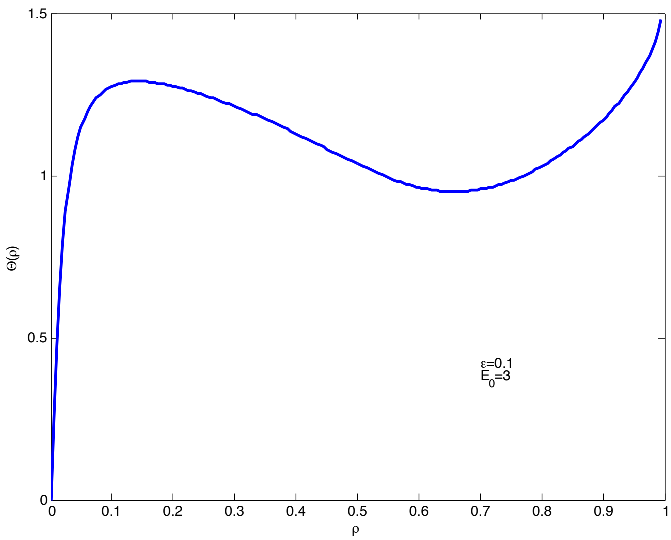

Figure 2.12: Map $\Theta(\rho)$ for the potential given by Eq. (2.11.10), with $\varepsilon = 0.1$ and $E_0 = V^2/2 = 3$.

- Even when Φ'' is nonnegative, the mapping can be non monotonic if Eq. (2.11.8) fails, an example being

$$\Phi(q) = \begin{cases} \frac{\delta^{k+2}}{k|q|^k} + \delta - \delta^2(1+\frac{1}{k}) & 0 < |q| < \delta \\ \delta(1-|q|) & \delta \le |q| < 1 \,, \\ 0 & |q| \ge 1 \end{cases} \tag{2.11.11}$$

We checked numerically the non monotonicity of $\Theta(\rho)$ in the case $\delta = 0.1, k = 4$ (see Fig. 2.13). Another example similar to the previous one but with continuous derivative in $|q| = 1$ can be constructed again by using a smooth junction.

4) Any time the condition $\frac{d\Theta}{d\rho} > 0$ is violated, we have a singularity of the cross–section, even though single–valued. In particular, if $\Phi'(1) = 0$ (i.e. the force is smooth) and $\Theta(\rho)$ is strictly monotonic, we still have a divergence of $\frac{d\rho}{d\Theta}$ for Θ near to $\pi/2$ ($\rho = 1$), that is

$$\Big|\Big| \sin(2\Theta)\sigma_\Phi \Big|\Big|_\infty = +\infty \,. \tag{2.11.12}$$

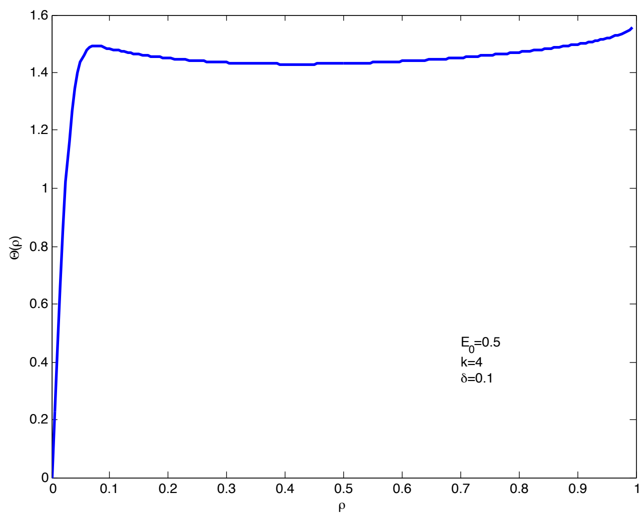

Figure 2.13: Map $\Theta(\rho)$ for the potential given by Eq. (2.11.11), with $\delta = 0.1$, $k = 4$ and $E_0 = V^2/2 = 0.5$.

Bibliography

[1] L. Boltzmann. *Lectures on gas theory.* English edition annotated by S. Brush, University of California Press, Berkeley, 1964 (reprint).

[2] C. Cercignani. On the Boltzmann equation for rigid spheres. *Transport Theory and Stat. Phys.* **2**, 211–225, 1972.

[3] C. Cercignani, R. Illner and M. Pulvirenti. The Mathematical Theory of Dilute Gases. *Applied Mathematical Sciences* **106**, Springer–Verlag, New York, 1994.

[4] L. Desvillettes and M. Pulvirenti. The linear Boltzmann equation for long–range forces: a derivation from particle systems. *Models Methods Appl. Sci.* **9**, 1123–1145, 1999.

[5] I. Gallagher, L. Saint Raymond and B. Texier. From Newton to Boltzmann: the case of short–range potentials. *arXiv: 1208.5753v1 [math.AP]*, 28 Aug 2012.

[6] H. Grad. Principles of the kinetic theory of gases. S. Flügge ed. *Handbuch der Physik* **12**, 205–294, 1958.

[7] R. Illner and M. Pulvirenti. Global Validity of the Boltzmann equation for a Two–Dimensional Rare Gas in the Vacuum. *Comm. Math. Phys.* **105**, 189–203, 1986.

[8] R. Illner and M. Pulvirenti. Global Validity of the Boltzmann equation for a Two– and Three–Dimensional Rare Gas in Vacuum: Erratum and Improved Result. *Comm. Math. Phys.* **121**, 143–146, 1989.

[9] F. King. *BBGKY Hierarchy for Positive Potentials.* Ph.D. Thesis, Department of Mathematics, Univ. California, Berkeley, 1975.

[10] O. E. Lanford. Time evolution of large classical systems. In "Dynamical systems, theory and applications", *Lecture Notes in Physics*, ed. J. Moser, **38**, 1–111, Springer–Verlag, Berlin, 1975.

[11] C. Marchioro, A. Pellegrinotti, E. Presutti and M. Pulvirenti. On the dynamics of particles in a bounded region: A measure theoretical approach. *J. Math. Phys.* **17**, 647, 1976.

[12] J. C. Maxwell. On the Dynamical Theory of Gases. *Philos. Trans. Roy. Soc. London Ser. A 157*, 49–88, 1867.

[13] J. C. Maxwell. *The Scientific Letters and Papers of James Clerk Maxwell: 1862–1873*, **2**. Cambridge University Press, 1995.

[14] D. Ruelle. *Statistical Mechanics. Rigorous results.* W. A. Benjamin Inc., New York, 1969.

[15] Y. Sone and K. Aoki. *Molecular Gas Dynamics.* Asakura-Shoten, Tokyo, 1994.

[16] H. Spohn. Boltzmann equation and Boltzmann hierarchy. In "Kinetic Theories and the Boltzmann equation", *Lecture Notes in Mathematics* **1048**, 207–220, ed. C. Cercignani, Springer–Verlag, Berlin, 1984.

[17] H. Spohn. On the Integrated Form of the BBGKY Hierarchy for Hard Sphereds. *arXiv: 0605068v1 [math-ph]*, 25 May 2006, 1985.

[18] H. Spohn. Large Scale Dynamics of Interacting Particles. *Texts and Monographs in Physics*, Springer–Verlag, Heidelberg, 1991.

[19] K. Uchiyama. Derivation of the Boltzmann equation from particle dynamics. *Hiroshima Math. J.* **18**, 245–297, 1988.

[20] S. Ukai. The Boltzmann–Grad limit and Cauchy–Kovalevskaya theorem. *Japan J. Indust. Appl. Math.* **18**, 383–392, 2001.

Chapter 3

Weak–coupling limit

In the present Chapter we present the paper [BPS].

3.1 From particle systems to the Landau equation: a consistency result

Abstract.
We consider a system of N classical particles, interacting via a smooth, short-range potential, in a weak-coupling regime. This means that N tends to infinity when the interaction is suitably rescaled. The j-particle marginals, which obey to the usual BBGKY hierarchy, are decomposed into two contributions: one small but strongly oscillating, the other hopefully smooth. Eliminating the first, we arrive to establish the dynamical problem in term of a new hierarchy (for the smooth part) involving a memory term. We show that the first order correction to the free flow converges, as $N \to \infty$, to the corresponding term associated to the Landau equation. We also show the related propagation of chaos.

3.2 Introduction

Lev Landau in 1936 proposed a kinetic equation, usually called Fokker-Planck-Landau equation (simply Landau equation in the sequel) which is a diffusion with friction in velocity, suitable to describe the behavior of a weakly interacting gas, in particular a Coulomb gas in a regime where the grazing collisions are dominant.

Roughly speaking the Landau's argument was to take the Boltzmann equation with Coulomb cross-section and (cutting-off short and long distances) apply the Taylor expansion to the collision operator. The result is a degenerate elliptic operator acting on the velocity space (see [18] and the original publication of Landau [17]). The full Taylor expansion of the Boltzmann collision integral for arbitrary intermolecular forces was studied in [6] and a formal generalization of Landau collision integral to arbitrary scattering cross-section was proposed there. A more precise asymptotics in the Coulomb case was also studied in [8].

The Landau equation for the one particle distribution $f(x,v,t)$, where $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ and $t \in \mathbb{R}_+$ denote position, velocity and time respectively, reads as

$$(\partial_t + v \cdot \nabla_x) f = Q_L(f,f) \tag{3.2.1}$$

with the collision operator Q_L given by:

$$Q_L(f,f)(v) = \int dv_1 \nabla_v \left[a(v-v_1) \left(\nabla_v - \nabla_{v_1} \right) f(v) f(v_1) \right]. \tag{3.2.2}$$

Here x plays the role of a parameter and hence its dependence is omitted. Moreover the matrix $a(w)$ has the form

$$a(w) = \frac{A}{|w|} \frac{(|w|^2 Id - w \otimes w)}{|w|^2}, \tag{3.2.3}$$

where $A > 0$ is a suitable constant.

Note that the Landau equation possesses all the properties known for the Boltzmann equation, namely the mass, momentum and energy conservation and the H-theorem. Actually the homogeneous Landau equation can be rigorously derived in the grazing collision limit of the homogeneous Boltzmann equation by a suitable rescaling of the cross-section.

In particular, in [1] the authors show that, under suitable assumptions on the cross–section, the diffusion Landau equation (3.2.1) can indeed be derived. The diffusion operator is the form (3.2.2) but with a matrix a replaced by

$$\alpha(|w|) \frac{(|w|^2 Id - w \otimes w)}{|w|^2},$$

with α a smooth function. Next in [13] and [23] steps forward were performed to arrive to cover the case $\alpha(|w|) \approx \frac{1}{|w|^\nu}$ for small $|w|$, with $\nu < 1$.

The case of the matrix (3.2.3) was treated in [24]. It is worth to underline that the initial value problem for the homogeneous Landau equation is strongly simplified for the case $\alpha(|w|) \approx \frac{1}{|w|^\nu}$, with $\nu < 1$ (see [3] and [4]), while for the matrix (3.2.3) we have a weak existence theorem obtained by compactness arguments based on the entropy production control [24]. Moreover, for the inhomogeneous case, we have existence and uniqueness of strong solutions for data sufficiently close to a Maxwellian [14]. This is the only existence and uniqueness result we are aware.

A natural question is to see whether the Landau equation can be directly derived, under a suitable scaling limit, from a particle system as it is the case of the Boltzmann equation. In fact one can see ([2], see also [22] and [21]), at a formal level, that the Landau equation is expected to be valid for a weakly interacting dense gas. The precise statement and scaling (called weak-coupling limit) will be presented and discussed in the next Section. The formal analysis gives indeed the Landau equation (3.2.1) with matrix (3.2.3). The two-body interaction potential ϕ is assumed smooth, spherically symmetric, and the constant A is given by:

$$A = \frac{1}{8\pi} \int_0^{+\infty} dr \, r^3 \hat{\phi}(r)^2, \tag{3.2.4}$$

where $\hat{\phi}(|k|) = \int dx \, \phi(|x|) e^{-ik\cdot x}$.

Note that we find the Landau equation with matrix (3.2.3), which is not related to the Coulomb potential, but arises even though the potential is smooth and short-range. This fact was first established by N.N. Bogolyubov in 1946 [20].

In the present paper we want to start the rigorous analysis of the weak-coupling limit for an Hamiltonian particle system. Our result is very preliminary. We first decompose the j-particle marginals into two terms, one hopefully smooth and the other strongly oscillating, but small. Eliminating this last term from the equations (with a procedure similar to that proposed by Zwanzig [26]) we find an equation with memory, which we can handled up to the first order in time. We show that this contribution agrees with the corresponding one arising from the Landau equation. Roughly speaking we present a rigorous derivation of the Landau equation at time zero.

It is well known that the situation for the Boltzmann equation is better, namely we are able to derive such a kinetic equation for a short time [16] (see also [7] for additional comments and results) in the low-density (or Boltmann-Grad) limit.

Note that the linear case, namely a single particle in a random potential under the weak-coupling limit, is well understood, see [11] and references quoted therein.

Our analysis deals with the nonlinear problem but our techniques could apply as well to the linear case. We think that, while we can easily obtain the same consistency result presented here, it seems very difficult to go further. In [11] and related references, it is crucial the use of probabilistic tools which seems more efficient compared with the hierarchical approaches. In contrast it is very difficult to implement the ideas working for the linear case to the present problem.

Finally we want to mention that the same problem of characterizing the weak-coupling limit of particle systems, arises also in a quantum mechanical context. In this case the quantity which we are interested in is the Wigner transform [25] which is a way to describe a quantum state as a function in the classical phase space. In contrast with the classical case, we expect that the Wigner transform approaches, in the weak-coupling limit, the solution of a suitable Boltzmann equation, with a corrections due to the statistics, whenever taken in explicit consideration. We quote [15], [3], [12], [4], [5] for the few results in this direction and [19] and references quoted therein, for the Boltzmann description of wave dynamics in the weak-coupling limit.

3.3 Weak-coupling limit for classical systems

We consider a classical system of N identical particles of unit mass in the whole space. Positions and velocities are denoted by the vectors $Q_N = \{q_1 \dots q_N\}$ and $V_N = \{v_1 \dots v_N\}$ respectively. The particles interact via a spherically symmetric, smooth potential of finite range $\phi : \mathbb{R}^3 \to \mathbb{R}$, namely $\phi(x) = 0$ if $|x| > r$ for some positive r. In the following we assume units for which $r = 1$.

The Newton equations read as:

$$\frac{d}{d\tau} q_i = v_i \qquad \frac{d}{d\tau} v_i = \sum_{\substack{j=1\dots N: \\ j\neq i}} F(q_i - q_j). \tag{3.3.1}$$

Here $F = -\nabla\phi$ denotes the interparticle (conservative) force, and τ is the time.

Let $\varepsilon > 0$ be a small parameter denoting the ratio between the macroscopic and microscopic space-time unities.

We are interested in a situation where the number of particles N is very large and the interaction strength quite moderate. The system has a unitary density so that we assume $N = \varepsilon^{-3}$. In addition we look for a reduced or macroscopic description of the system. Namely if q and τ refer to the system seen in a microscopic scale, we rescale eq.n (3.3.1) in terms of the macroscopic variables

$$x = \varepsilon q \qquad t = \varepsilon\tau$$

whenever the physical variables of interest are varying on such scales and are almost constant on the microscopic scales.

Remembering that we want to describe weakly interacting systems, we also rescale the potential according to:

$$\phi \to \sqrt{\varepsilon}\phi, \tag{3.3.2}$$

so that system (3.3.1), in terms of the (x, t) variables, becomes:

$$\frac{d}{dt} x_i = v_i \qquad \frac{d}{dt} v_i = -\frac{1}{\sqrt{\varepsilon}} \sum_{\substack{j=1\dots N: \\ j\neq i}} \nabla\phi(\frac{x_i - x_j}{\varepsilon}) = \frac{1}{\sqrt{\varepsilon}} \sum_{\substack{j=1\dots N: \\ j\neq i}} F(\frac{x_i - x_j}{\varepsilon}). \tag{3.3.3}$$

Note that the velocities are automatically unscaled.

A statistical description of the above system passes through the introduction of a probability distribution on the phase space of the system. Let $W^N = W^N(X_N, V_N)$ be a symmetric (in the exchange of variables) probability distribution. Here (X_N, V_N) denote the set of positions and velocities:

$$X_N = \{x_1 \dots x_N\} \quad V_N = \{v_1 \dots v_N\}, \quad x_i \in \mathbb{R}^3, v_i \in \mathbb{R}^3.$$

Then from eq.ns (3.3.3) we obtain the following Liouville equation

$$(\partial_t + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i}) W^N(X_N, V_N) = \frac{1}{\sqrt{\varepsilon}} \big(T_N^\varepsilon W^N\big)(X_N, V_N). \tag{3.3.4}$$

Here we have introduced the operator

$$(T^\varepsilon_N W^N)(X_N, V_N) = \sum_{0<k<\ell\le N} (T^\varepsilon_{k,\ell} W^N)(X_N, V_N), \tag{3.3.5}$$

with

$$T^\varepsilon_{k,\ell} W^N = \nabla\phi(\frac{x_k - x_\ell}{\varepsilon}) \cdot (\nabla_{v_k} - \nabla_{v_\ell}) W^N. \tag{3.3.6}$$

To investigate the limit $\varepsilon \to 0$ it is convenient to introduce the BBKGY hierarchy for the j-particle distributions defined as

$$f^N_j(X_j, V_j) = \int dx_{j+1} \dots \int dx_N \int dv_{j+1} \dots \int dv_N \tag{3.3.7}$$
$$W^N(X_j, x_{j+1} \dots x_N; V_j, v_{j+1} \dots v_N)$$

for $j = 1 \dots, N-1$. Obviously we set $f^N_N = W^N$. Note that BBGKY stands for Bogolyubov, Born, Green, Kirkwood and Yvon, the names of physicists who introduced independently this system of equations (see e.g. [2]).

Such a hierarchy is obtained by means of a partial integration of the Liouville equation (3.3.4) and standard manipulations. The result is (for $1 \le j \le N$):

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k}) f^N_j = \frac{1}{\sqrt{\varepsilon}} T^\varepsilon_j f^N_j + \frac{N-j}{\sqrt{\varepsilon}} C^\varepsilon_{j+1} f^N_{j+1} \tag{3.3.8}$$

for $1 \le j \le N$.

We set

$$f^N_j = 0, \quad \text{for} \quad j > N, \quad \text{and} \quad f^N_N = W^N.$$

The operator C^ε_{j+1} is defined as:

$$C^\varepsilon_{j+1} = \sum_{k=1}^{j} C^\varepsilon_{k,j+1}\ , \tag{3.3.9}$$

and

$$C^\varepsilon_{k,j+1} f_{j+1}(x_1 \dots x_j; v_1 \dots v_j) = \tag{3.3.10}$$
$$-\int dx_{j+1} \int dv_{j+1} F\left(\frac{x_k - x_{j+1}}{\varepsilon}\right) \cdot \nabla_{v_k} f_{j+1}(x_1, x_2, \dots, x_{j+1}; v_1, \dots, v_{j+1}).$$

$C^\varepsilon_{k,j+1}$ describes the interaction of particle k, belonging to the j-particle subsystem, with a particle outside the subsystem, conventionally denoted by the number $j+1$ (this numbering uses the fact that all the particles are identical).

We finally fix the initial value $\{f^0_j\}^N_{j=1}$ of the solution $\{f^N_j(t)\}^N_{j=1}$ assuming that $\{f^0_j\}^N_{j=1}$ is factorized, that is, for all $j = 1, \dots N$

$$f^0_j = f_0^{\otimes j}, \tag{3.3.11}$$

where f_0 is a given one-particle distribution function. This means that the state of any pair of particles is statistically uncorrelated at time zero. Of course such a statistical independence is destroyed at time $t > 0$ because dynamics creates correlations and eq.n (3.3.8) shows that the time evolution of f_1^N is determined by the knowledge of f_2^N which turns out to be dependent on f_3^N and so on. However, since the interaction between two given particles is going to vanish in the limit $\varepsilon \to 0$, we can hope that such statistical independence is recovered in the same limit. Therefore we expect that when $\varepsilon \to 0$ the one-particle distribution function f_1^N converges to the solution of a suitable nonlinear kinetic equation f, which we are going to investigate.

If we expand $f_j^N(t)$ as a perturbation of the free flow $S(t)$ defined as

$$(S(t)f_j)(X_j, V_j) = f_j(X_j - V_j t, V_j), \tag{3.3.12}$$

we find

$$\begin{aligned} f_j^N(t) =& S(t)f_j^0 + \frac{N-j}{\sqrt{\varepsilon}} \int_0^t S(t-t_1) C_{j+1}^\varepsilon f_{j+1}^N(t_1) dt_1 + \\ & \frac{1}{\sqrt{\varepsilon}} \int_0^t S(t-t_1) T_j^\varepsilon f_j^N(t_1) dt_1. \end{aligned} \tag{3.3.13}$$

We now try to keep information on the limit behavior of $f_j^N(t)$. Assuming for the moment that the time evolved j-particle distributions $f_j^N(t)$ are smooth (in the sense that the first and second derivatives are uniformly bounded in ε), then

$$\begin{aligned} & C_{j+1}^\varepsilon f_{j+1}^N(X_j; V_j; t_1) = \\ & -\varepsilon^3 \sum_{k=1}^j \int dr \int dv_{j+1} F(r) \cdot \nabla_{v_k} f_{j+1}(X_j, x_k - \varepsilon r; V_j, v_{j+1}, t_1). \end{aligned} \tag{3.3.14}$$

Because of the identity

$$\int dr F(r) = 0, \tag{3.3.15}$$

we find that

$$C_{j+1}^\varepsilon f_{j+1}^N(X_j; V_j; t_1) = O(\varepsilon^4) \tag{3.3.16}$$

provided that $D_v^2 f_{j+1}^N$ is uniformly bounded. Since

$$\frac{N-j}{\sqrt{\varepsilon}} = O(\varepsilon^{-\frac{7}{2}})$$

we see that the second term in the right hand side of (3.3.13) does not give any contribution in the limit.

Moreover

$$\begin{aligned} & \int_0^t S(t-t_1) T_j^\varepsilon f_j^N(t_1) dt_1 = \\ & \sum_{i \neq k} \int_0^t dt_1 F\left(\frac{(x_i - x_k) - (v_i - v_k)(t - t_1)}{\varepsilon} \right) \tilde{f}(X_j, V_j; t_1) \end{aligned} \tag{3.3.17}$$

where $\tilde{f}$ is a smooth function. We note that the time integral in (3.3.17) is $O(\varepsilon)$ because $F \neq 0$ only for times in an interval of length $O(\varepsilon)$. Therefore f_j^N cannot be smooth since we expect a nontrivial limit.

In order to look for a (nontrivial) kinetic equation, we can conjecture that

$$f_j^N = g_j^N + \gamma_j^N \tag{3.3.18}$$

where g_j^N is the main part of f_j^N and is smooth, while γ_j^N is small, but strongly oscillating. We operate this decomposition according to the following equations which define g_j^N and γ_j^N:

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k}) g_j^N = \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^{\varepsilon} g_{j+1}^N + \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^{\varepsilon} \gamma_{j+1}^N \tag{3.3.19}$$

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k}) \gamma_j^N = \frac{1}{\sqrt{\varepsilon}} T_j^{\varepsilon} \gamma_j^N + \frac{1}{\sqrt{\varepsilon}} T_j^{\varepsilon} g_j^N, \tag{3.3.20}$$

with initial data

$$g_j^N(X_j, V_j, 0) = f_j^0(X_j, V_j), \quad \gamma_j^N(X_j, V_j) = 0. \tag{3.3.21}$$

Note that $\gamma_1^N = 0$ since $T_1^{\varepsilon} = 0$.

The remarkable fact of this decomposition is that γ can be eliminated. Indeed, let

$$(X_j(t), V_j(t)) = (\{x_1(t) \dots x_j(t), v_1(t) \dots v_j(t)\})$$

be the solution of the j-particle flow (in macro variables)

$$\frac{d}{dt} x_i = v_i \qquad \frac{d}{dt} v_i = -\frac{1}{\sqrt{\varepsilon}} \sum_{\substack{k=1\dots j: \\ k \neq i}} \nabla \phi \left(\frac{x_i - x_k}{\varepsilon} \right), \tag{3.3.22}$$

with initial datum $(X_j, V_j) = (\{x_1 \dots x_j, v_1 \dots v_j\})$. Denote by $U_j(t)$ the operator

$$U_j(t) f(X_j, V_j) = \exp\{t(-\sum_i v_i \cdot \nabla_{x_i} + \frac{1}{\sqrt{\varepsilon}} T_j)\} f(X_j, V_j) = f(X_j(-t), V_j(-t)), \tag{3.3.23}$$

then eq.n (3.3.20) can be solved:

$$\gamma_j^N(t) = \int_0^t ds U_j(s) \frac{1}{\sqrt{\varepsilon}} T_j g_j^N(t-s). \tag{3.3.24}$$

Explicitly

$$\gamma_j^N(X_j, V_j, t) = -\frac{1}{\sqrt{\varepsilon}} \int_0^t ds \sum_{1 \leq i < k \leq j} F\left(\frac{x_i(-s) - x_k(-s)}{\varepsilon} \right) \cdot [(\nabla_{v_i} - \nabla_{v_k}) g_j^N](X_j(-s), V_j(-s); t-s). \tag{3.3.25}$$

Inserting (3.3.24) in (3.3.19) we finally arrive to a closed hierarchy for $\{g_j^N\}_{j=1}^N$. Obviously we pay the price of a memory term given by the time integral in (3.3.24) or in (3.3.25).

We write the hierarchy in integral form. Then

$$\begin{aligned} g_j^N(t) =& S(t) f_j^0 + \frac{N-j}{\sqrt{\varepsilon}} \int_0^t S(t-\tau) C_{j+1}^{\varepsilon} g_{j+1}^N(\tau) d\tau \\ &+ \frac{N-j}{\sqrt{\varepsilon}} \int_0^t S(t-\tau) C_{j+1}^{\varepsilon} \gamma_{j+1}^N(\tau) d\tau \\ =& S(t) f_j^0 + \frac{N-j}{\sqrt{\varepsilon}} \int_0^t S(t-\tau) C_{j+1}^{\varepsilon} g_{j+1}^N(\tau) d\tau \\ &+ \frac{N-j}{\varepsilon} \int_0^t d\tau \int_0^{\tau} d\sigma S(t-\tau) C_{j+1}^{\varepsilon} U_{j+1}(\tau-\sigma) T_{j+1} g_{j+1}^N(\sigma). \end{aligned} \tag{3.3.26}$$

Remark 1. *Why do we expect that γ_j^N strongly oscillates? Let us try to control the first derivatives of $h(X_j, V_j, t) = U_j(t) h_0(X_j, V_j) = h_0(X_j(-t), V_j(-t))$ for a given smooth function h_0. Then*

$$\frac{\partial h(X_j, V_j, t)}{\partial x_i^{\alpha}} = \sum_{k,\beta} \left(\frac{\partial h_0}{\partial x_k^{\beta}} (X_j(-t), V_j(-t)) \frac{\partial x_k^{\beta}(-t)}{\partial x_i^{\alpha}} + \frac{\partial h_0}{\partial v_k^{\beta}} (X_j(-t), V_j(-t)) \frac{\partial v_k^{\beta}(-t)}{\partial x_i^{\alpha}} \right)$$

and analogous formula for $\frac{\partial h(t)}{\partial v_i^{\alpha}}$. Here we are using Greek indices for the components of x_i and v_i. To estimate quantities like $\frac{\partial x_k^{\beta}(-t)}{\partial x_i^{\alpha}}, \frac{\partial x_k^{\beta}(-t)}{\partial v_i^{\alpha}}, \frac{\partial v_k^{\beta}(-t)}{\partial x_i^{\alpha}}, \frac{\partial v_k^{\beta}(-t)}{\partial v_i^{\alpha}}$ we use eq.n (3.3.22) and find (changing $-t \to t$)

$$\frac{d}{dt} \frac{\partial x_k^{\beta}(t)}{\partial x_i^{\alpha}} = \frac{\partial v_k^{\beta}(t)}{\partial x_i^{\alpha}}, \tag{3.3.27}$$

$$\frac{d}{dt} \frac{\partial v_k^{\beta}(t)}{\partial x_i^{\alpha}} = \frac{1}{\varepsilon^{3/2}} \sum_{\substack{r=1\dots j:\\ r\neq k}} \frac{\partial F^{\beta}}{\partial x_k^{\gamma}} \left(\frac{x_k(t) - x_r(t)}{\varepsilon}\right) \left(\frac{\partial x_k^{\gamma}(t)}{\partial x_i^{\alpha}} - \frac{\partial x_r^{\gamma}(t)}{\partial x_i^{\alpha}} \right). \tag{3.3.28}$$

Integrating eq.ns (3.3.27) *and* (3.3.28) *in time, we arrive, by using the Gronwall lemma, to*

$$\left| \frac{\partial v_k^{\beta}(t)}{\partial x_i^{\alpha}} \right| \leq C \exp\left(\frac{C \tau_c}{\varepsilon^{3/2}} \right)$$

where τ_c is the scattering time, namely the time interval for which $|x_k(t) - x_r(t)| \leq \varepsilon$. Now, even though $\tau_c = O(\varepsilon)$ (neglecting small relative velocities), it seems difficult to get something better than a bound like $\exp(\frac{C}{\sqrt{\varepsilon}})$.

In conclusion we expect that the first derivatives of $h(t)$ are $O(\exp(\frac{1}{\sqrt{\varepsilon}}))$. Looking at eq.n (3.3.24) *we expect for γ the same behavior. In contrast, the action of the operator C_j is regularizing (althoug we are not able to prove this) so that we expect g to be smooth.*

On the other hand γ_j^N is also expected to be small, in some sense. Indeed by taking the scalar product of (3.3.24) by a smooth function u, we find

$$
\begin{aligned}
|(u,\gamma_j^N(t))| \leq & \frac{1}{\sqrt{\varepsilon}}\int_0^t ds \|U_j(-s)u\|_{L^\infty}\|T_j^\varepsilon g_j^N(t-s)\|_{L^1} \\
& \leq \varepsilon^{5/2}\frac{j(j-1)}{2}\|u\|_{L^\infty}\int_0^t ds\int dx_1\int dx_3\ldots\int dV_j\int dr|F(r)| \\
& \quad |(\nabla_{v_1}-\nabla_{v_2})g_j^N(x_1,x_1+\varepsilon r,x_3\ldots,V_j;t-s)|.
\end{aligned}
$$

Therefore this term is vanishing provided that g^N is sufficiently smooth (uniformly in ε).

A rigorous analysis of the limit $N\to\infty$, $\varepsilon = N^{-(1/3)}$ seems to be very difficult. We expect that, in this limit, both $f_j^N(t)$ and $g_j^N(t)$ would converge to $f(t)^{\otimes j}$, where f solves the Landau equation stated in Introduction. We cannot prove it, but a first step in this direction is made in the following Sections.

3.4 Consistency

We consider eq.n (3.3.26) written in symbolic form as

$$
g_j = S(t)f_j^0 + A_{j+1}g_{j+1}, \tag{3.4.1}
$$

where all upper indices N are omitted for brevity. To solve these equations one can use the obvious iterative scheme

$$
g_j^0 = S(t)f_j^0, \quad g_j^{(n+1)} = S(t)f_j^0 + A_{j+1}g_{j+1}^{(n)}, \quad n=0,1,\ldots
$$

Our goal in this section is to prove that the equation for $g_1^{(1)}(t) = \tilde{g}_1^N(t)$ is consistent with the Landau equation. Thus we replace (3.3.26) by its first approximation:

$$
\begin{aligned}
\tilde{g}_j^N(t) =& S(t)f_j^0 + \frac{N-j}{\sqrt{\varepsilon}}\int_0^t S(t-\tau)C_{j+1}^\varepsilon S(\tau)f_{j+1}^0 d\tau && (3.4.2)\\
&+ \frac{N-j}{\sqrt{\varepsilon}}\int_0^t S(t-\tau)C_{j+1}^\varepsilon \tilde{\gamma}_{j+1}^N(\tau)d\tau \\
=& S(t)f_j^0 + \frac{N-j}{\sqrt{\varepsilon}}\int_0^t S(t-\tau)C_{j+1}^\varepsilon S(\tau)g_{j+1}^N d\tau && (3.4.3)\\
&+ \frac{N-j}{\varepsilon}\int_0^t d\tau\int_0^\tau d\sigma S(t-\tau)C_{j+1}^\varepsilon U_{j+1}(\tau-\sigma)T_{j+1}S(\sigma)f_{j+1}^0.
\end{aligned}
$$

Here we set

$$
\begin{aligned}
\tilde{\gamma}_j^N(X_j,V_j,\tau) =& \frac{1}{\sqrt{\varepsilon}}\int_0^\tau d\sigma U_j(\tau-\sigma)T_jS(\sigma)f_{j+1}^0 && (3.4.4)\\
=& -\frac{1}{\sqrt{\varepsilon}}\int_0^t ds\sum_{1\leq i<k\leq j}F(\frac{x_i(-s)-x_k(-s)}{\varepsilon})\cdot && (3.4.5)\\
& \cdot[(\nabla_{v_i}-\nabla_{v_k})S(\tau-s)f_j^0](X_j(-s),V_j(-s)).
\end{aligned}
$$

We note that $\tilde{\gamma}_j^N$ can be explicitly computed.

Lemma 1. *We have*

$$
\begin{aligned}
\tilde{\gamma}_j^N(X_j,V_j,t) &= (U_j(t)f_j^0 - S(t)f_j^0)(X_j,V_j)\\
&= f_j^0(X_j(-t),V_j(-t)) - f_j^0(X_j - V_j t, V_j)\;. \qquad (3.4.6)
\end{aligned}
$$

Proof. Let $\mathcal{L}_0 = -\sum_i v_i \cdot \nabla_{x_i}$ be the free flow generator.
Then we compute

$$
\begin{aligned}
U_j(t)f_j^0 - S(t)f_j^0 =& \int_0^t ds \frac{d}{ds}[U_j(s)S(t-s)]f_j^0 \qquad (3.4.7)\\
=& \int_0^t ds [U_j(s)(\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}T_j)S(t-s)]f_j^0\\
&- \int_0^t ds [U_j(s)\mathcal{L}_0 S(t-s)]f_j^0\\
=& \tilde{\gamma}_j^N(t).
\end{aligned}
$$

□ □

For convenience of the reader we make explicit eq.n (3.4.2) in the case $j = 1$

$$
\begin{aligned}
\tilde{g}_1^N(t) =& S(t)f_0 + \frac{N-1}{\sqrt{\varepsilon}} \int_0^t S(t-\tau)C_2^\varepsilon S(\tau) f_2^0 d\tau \qquad (3.4.8)\\
&+ \frac{N-1}{\sqrt{\varepsilon}} \int_0^t S(t-\tau)C_2^\varepsilon \tilde{\gamma}_2^N(\tau) d\tau
\end{aligned}
$$

where, by Lemma 1,

$$
\begin{aligned}
\tilde{\gamma}_2^N(x_1,v_1,x_2,v_2,\tau) =& -\frac{1}{\sqrt{\varepsilon}} \int_0^\tau ds F(\frac{x_1(-s) - x_2(-s)}{\varepsilon})\cdot \qquad (3.4.9)\\
&\cdot [(\nabla_{v_1} - \nabla_{v_2})S(\tau - s)f_2^0](X_2(-s),V_2(-s))\\
=& \big[f_2^0(X_2(-\tau),V_2(-\tau)) - f_2^0(X_2 - V_2\tau, V_2)\big].
\end{aligned}
$$

The first result of the present paper is summarized in the following Theorem.

Theorem 3. *Suppose $f_0 \in C_0^3(\mathbb{R}^3 \times \mathbb{R}^3)$ be the initial probability density satisfying:*

$$
|D^r f_0(x,v)| \le Ce^{-b|v|^2} \quad \textit{for} \qquad r = 0,1,2 \qquad (3.4.10)
$$

where D^r is any derivative of order r and $b > 0$. Assuming also that $\phi \in C^2(\mathbb{R}^3)$ and $\phi(x) = 0$ if $|x| > 1$. If (3.3.11) holds for $j = 1,2$, then

$$
\lim_{\varepsilon \to 0} \tilde{g}_1^N(t) = S(t)f_0 + \int_0^t d\tau S(t-\tau) Q_L(S(\tau)f_0, S(\tau)f_0), \qquad (3.4.11)
$$

$$
\lim_{\varepsilon \to 0} \tilde{\gamma}_1^N(t) = 0, \qquad (3.4.12)
$$

where $N\varepsilon^3 = 1$ and the above limits are considered in $\mathcal{D}'$.

Proof. Let $u \in \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3)$ be a test function. From now on we will denote by $(h_j, k_j) = \int dX_j \int dV_j\, h_j(X_j, V_j) k_j(X_j, V_j)$ the inner product. Then

$$(u, \tilde{g}_1^N(t)) = (u, S(t) f_0) + \frac{N-1}{\sqrt{\varepsilon}} \int_0^t (u, S(t-\tau) C_2^\varepsilon S(\tau) f_2^0) d\tau_1 + \int_0^t \mathcal{T}_\varepsilon(\tau) d\tau, \tag{3.4.13}$$

where

$$\begin{aligned}\mathcal{T}_\varepsilon(\tau) = -\frac{N-1}{\sqrt{\varepsilon}} \int dx_1 \int dx_2 \int dv_1 \int dv_2 (\nabla_{v_1} S(\tau - t) u(x_1, v_1)) \cdot \\ \cdot F\left(\frac{x_1 - x_2}{\varepsilon}\right) \tilde{\gamma}_2^N (x_1, x_2, v_1, v_2, \tau).\end{aligned} \tag{3.4.14}$$

We have already seen that the second term in the right hand side of (3.4.13) is vanishing. Therefore we have to evaluate the last term, namely $\int_0^t d\tau \mathcal{T}_\varepsilon(\tau)$. We split the term $\mathcal{T}_\varepsilon(\tau)$ into two terms

$$\mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon^{\leq} + \mathcal{T}_\varepsilon^{>} \tag{3.4.15}$$

where

$$\begin{aligned}\mathcal{T}_\varepsilon(\tau)^{>} = -\frac{N-1}{\sqrt{\varepsilon}} \int dx_1 \int dx_2 \int dv_1 \int_{|w| > a\varepsilon^{1/4}} dv_2 (\nabla_{v_1} S(\tau - t) u(x_1, v_1)) \cdot \\ \cdot F\left(\frac{x_1 - x_2}{\varepsilon}\right) \tilde{\gamma}_2^N (x_1, x_2, v_1, v_2, \tau)\end{aligned} \tag{3.4.16}$$

where $w = v_1 - v_2$ is the relative velocity and a is a number to be fixed later on. $\mathcal{T}_\varepsilon^{\leq}$ is defined accordingly.

The reason of this decomposition will be clear later on. For the moment we show that $\mathcal{T}_\varepsilon^{\leq}$ is negligible.

Lemma 2.

$$\mathcal{T}_\varepsilon^{\leq} = O(\varepsilon^{1/4}) \,. \tag{3.4.17}$$

Proof. By Lemma 1 we have that $\tilde{\gamma}_2^N$ is uniformly bounded. Moreover by the change of variables

$$x_2 = x_1 - \varepsilon r$$

we get

$$\begin{aligned}|\mathcal{T}_\varepsilon^{\leq}| \leq & C(N-1)\varepsilon^3 \frac{1}{\sqrt{\varepsilon}} \int dx_1 \int dv_1 |\nabla_{v_1} S(\tau - t) u(x_1, v_1)| \int dr |F(r)| \int_{|w| \leq a\varepsilon^{1/4}} dw \\ \leq & C\varepsilon^{1/4}.\end{aligned} \tag{3.4.18}$$

□ □

To evaluate $\mathcal{T}_\varepsilon^>$ we use (3.4.9) to write it as

$$\begin{aligned}\mathcal{T}_\varepsilon^>(\tau) =&(N-1)\varepsilon^3 \int dx_1 \int dr \int dv_1 \int_{|w|>a\varepsilon^{1/4}} dv_2 \\ &\frac{1}{\varepsilon}\int_0^\tau dsF_\alpha(r)F_\beta\left(\frac{x_1(-s)-x_2(-s)}{\varepsilon}\right)[h_\varepsilon(x_1,x_2,v_1,v_2,\tau,s)]_{\alpha,\beta},\end{aligned} \tag{3.4.19}$$

where $x_2 = x_1 - \varepsilon r$ and h_ε is the matrix

$$(h_\varepsilon)_{\alpha,\beta} = -(\nabla_{v_1}S(\tau-t)u(x_1,v_1))_\alpha[(\nabla_{v_1}-\nabla_{v_2})S(\tau-s)f_2^0]_\beta(X_2(-s),V_2(-s)), \quad \alpha,\beta=1,2,3.$$

The summation over repeated Greek indices is assumed here and below.
Here the flow $X_2(t) = (x_1(t), x_2(t))$ has initial conditions $(x_1, x_1-\varepsilon r)$. Scaling times we also find

$$\begin{aligned}\mathcal{T}_\varepsilon^>(\tau) =&(N-1)\varepsilon^3 \int dx_1 \int dr \int dv_1 \int_{|w|>a\varepsilon^{1/4}} dv_2 \\ &\int_0^{\tau/\varepsilon} dsF_\alpha(r)F_\beta\left(\frac{x_1(-\varepsilon s)-x_2(-\varepsilon s)}{\varepsilon}\right)[h_\varepsilon(x_1,x_2,v_1,v_2,\tau,\varepsilon s)]_{\alpha,\beta}\,.\end{aligned} \tag{3.4.20}$$

Let us introduce the function h which is the formal limit of h_ε, namely

$$h_{\alpha,\beta}(x_1,v_1,v_2,\tau) = -R_\alpha(x_1,v_1,\tau)[(\nabla_{v_1}-\nabla_{v_2})S(\tau)f_2^0(x_1,x_1,v_1,v_2)]_\beta, \tag{3.4.21}$$

where

$$R(x_1,v_1,\tau) = \nabla_{v_1}S(\tau-t)u(x_1,v_1). \tag{3.4.22}$$

We split $\mathcal{T}_\varepsilon^>$ into two terms

$$\mathcal{T}_\varepsilon^> = \mathcal{T}_1^> + \mathcal{T}_2^>$$

where

$$\begin{aligned}\mathcal{T}_1^>(\tau) = &(N-1)\varepsilon^3 \int dx_1 \int dr \int dv_1 \int_{|w|>a\varepsilon^{1/4}} dv_2 \\ &\int_0^{\frac{\tau}{\varepsilon}} dsF_\alpha(r)F_\beta\left(\frac{x_1(-\varepsilon s)-x_2(-\varepsilon s)}{\varepsilon}\right)h_{\alpha,\beta}(x_1,v_1,v_2,\tau)\end{aligned} \tag{3.4.23}$$

and

$$\begin{aligned}\mathcal{T}_2^>(\tau) = &(N-1)\varepsilon^3 \int dx_1 \int dr \int dv_1 \int_{|w|>a\varepsilon^{1/4}} dv_2 \\ &\int_0^{\frac{\tau}{\varepsilon}} dsF_\alpha(r)F_\beta\left(\frac{x_1(-\varepsilon s)-x_2(-\varepsilon s)}{\varepsilon}\right)(h_\varepsilon-h)_{\alpha,\beta}.\end{aligned} \tag{3.4.24}$$

We shall show that $\mathcal{T}_2^>(\tau)$ is vanishing while $\mathcal{T}_1^>(\tau)$ has the right behavior. In the evaluation of $\mathcal{T}_1^>(\tau)$ we note that h does not depend on s so that we have to evaluate the integral

$$\int_0^{\frac{\tau}{\varepsilon}} dsF\left(\frac{x_1(-\varepsilon s)-x_2(-\varepsilon s)}{\varepsilon}\right) = \frac{1}{\varepsilon}\int_0^\tau dsF\left(\frac{x_1(-s)-x_2(-s)}{\varepsilon}\right). \tag{3.4.25}$$

Indeed the integral (3.4.25) can be bounded when the interaction time of the two-particle system is $O(\varepsilon)$ and this is true only if the relative velocity is not too small (see Lemma 3 below). This explains why we did the decomposition (3.4.15).

Lemma 3. *Setting $w = v_1 - v_2$, suppose that*

$$|w| > a\varepsilon^{1/4} \tag{3.4.26}$$

where $a = 4\sqrt{\|F\|_{L^\infty}}$. Then, defining for any real number s

$$\Delta_\varepsilon = \{s||x_1(s) - x_2(s)| < \varepsilon\}, \tag{3.4.27}$$

we have

$$\text{meas}(\Delta_\varepsilon) \leq \frac{4\varepsilon}{|w|}. \tag{3.4.28}$$

Moreover, for $i = 1, 2$:

$$|v_i(\varepsilon s) - v_i| \leq C\frac{\sqrt{\varepsilon}}{|w|}. \tag{3.4.29}$$

Proof. Assuming first that $s > 0$, we pass in the coordinate system around the center of mass (at the origin) and denote by $\xi(t) = x_1(t) - x_2(t)$. Let $w = v_1 - v_2$ be the relative velocity and w_x its horizontal component. We assume that at time zero the particles are in the interaction disk (more precisely, they enter in the interaction disk at time $s = 0$) and fix the axis in such a way that w is horizontal and its x- component is positive, namely $w_x = |w|$. Let $\bar{t}$ be the first time for which

$$w_x(t) \leq \frac{|w|}{2}.$$

By the equation of motion

$$w_x(t) = |w| + \int_0^t \frac{2}{\sqrt{\varepsilon}} F_x\left(\frac{\xi(s)}{\varepsilon}\right) \tag{3.4.30}$$

we infer

$$\frac{|w|}{2} \geq |w| - \frac{2}{\sqrt{\varepsilon}}\|F\|_{L^\infty}\bar{t}$$

from which

$$\bar{t} \geq \frac{\sqrt{\varepsilon}|w|}{4\|F\|_{L^\infty}}. \tag{3.4.31}$$

In the time interval $[0, \bar{t}]$ we have $w_x \geq \frac{|w|}{2}$ and the horizontal displacement is (under assumption (3.4.26)) larger than

$$\frac{|w|}{2}\bar{t} \geq 2\varepsilon, \tag{3.4.32}$$

since the diameter 2ε is a maximal path inside the sphere, independent of the initial point. This implies that, when $|\xi(t)| < \varepsilon$, then $|w(t)| > |w(0)|/2$ and hence

$$\text{meas}(\Delta_\varepsilon) \leq \frac{4\varepsilon}{|w|}. \tag{3.4.33}$$

Moreover

$$v_1(\varepsilon s) = v_1 + \int_0^{\varepsilon s} \frac{1}{\sqrt{\varepsilon}} F\left(\frac{x_1(\sigma) - x_2(\sigma)}{\varepsilon}\right) d\sigma \tag{3.4.34}$$

from which

$$|v_1(\varepsilon s) - v_1| \le C\frac{\sqrt{\varepsilon}}{|w|}. \tag{3.4.35}$$

The case $s < 0$ reduces to the case $s > 0$ by changing the initial velocities to $v_i(0) = -v_i$ for $i = 1, 2$. This completes the proof. □ □

Note that

$$\frac{x_1(-\varepsilon s) - x_2(-\varepsilon s)}{\varepsilon} = r - ws + \frac{1}{\varepsilon}\int_0^{-\varepsilon s} d\sigma[(v_1(\sigma) - v_1) - (v_2(\sigma) - v_2)] \tag{3.4.36}$$

thus, by Lemma 3,

$$\left|\frac{x_1(-\varepsilon s) - x_2(-\varepsilon s)}{\varepsilon} - (r - ws)\right| \le Cs\frac{\sqrt{\varepsilon}}{|w|}. \tag{3.4.37}$$

The integral (3.4.23) reads

$$\begin{aligned}\mathcal{T}_1^{>}(\tau) = (N-1)\varepsilon^3 \int dx_1 \int dr \int dv_1 \int_{|w|>a\varepsilon^{1/4}} dv_2 \\ \int_0^{\frac{\tau}{\varepsilon}} ds F_\alpha(r) F_\beta(r - ws) h_{\alpha,\beta}(x_1, v_1, v_2, \tau) + E\end{aligned} \tag{3.4.38}$$

where the error term E is given by

$$\begin{aligned}E = (N-1)\varepsilon^3 \int dx_1 \int dr \int dv_1 \int_{|w|>a\varepsilon^{1/4}} dv_2 \\ \int_0^{\frac{\tau}{\varepsilon}} ds F_\alpha(r) \left[F_\beta\left(\frac{x_1(-\varepsilon s) - x_2(-\varepsilon s)}{\varepsilon}\right) - F_\beta(r - ws)\right] h_{\alpha,\beta}(x_1, v_1, v_2, \tau)\end{aligned} \tag{3.4.39}$$

It is clear from the proof of Lemma 3 that $|x_1(-\varepsilon s) - x_2(-\varepsilon s)| \ge \varepsilon$ if $s \ge 4/|w|$ (see (3.4.32)). On the other hand, $|r - ws| \ge 1$ if $s \ge 2/|w|$, provided $|r| \le 1$. Hence,

$$\begin{aligned}|E| \le & C\sqrt{\varepsilon} \int dx_1 \int dv_1 \int_{|w|>a\varepsilon^{1/4}} dv_2 \frac{1}{|w|} \int_0^{\frac{4}{|w|}} s ds |h(x_1, v_1, v_2, \tau)| \\ \le & C\sqrt{\varepsilon} \int dx_1 \int dv_1 \int_{|w|>a\varepsilon^{1/4}} dv_2 \frac{1}{|w|^3} |h(x_1, v_1, v_2, \tau)| \\ \le & C\sqrt{\varepsilon} |\log \varepsilon|.\end{aligned} \tag{3.4.40}$$

In the last step we estimated

$$\begin{aligned}|h(x_1, v_1, v_2, \tau) \le & C|f_0(x_1 - v_2\tau, v_2)(\nabla_{v_1} - \tau\nabla_{x_1}) f_0(x_1 - v_1\tau, v_1)| \\ & + |f_0(x_1 - v_1\tau, v_1)(\nabla_{v_2} - \tau\nabla_{x_2}) f_0(x_1 - v_2\tau, v_2)| \\ \le & Ce^{-b(|v_1|^2 + |v_2|^2)}.\end{aligned} \tag{3.4.41}$$

Lemma 4. *For all $w \neq 0$,*

$$\lim_{\varepsilon \to 0} \int dr \int_0^{+\tau/\varepsilon} ds F_\alpha(r) F_\beta(r - ws) = \frac{1}{2} \lim_{\varepsilon \to 0} \int dr \int_{-\tau/\varepsilon}^{+\tau/\varepsilon} ds F_\alpha(r) F_\beta(r - ws) = a(w)_{\alpha,\beta} \tag{3.4.42}$$

where

$$a(w)_{\alpha,\beta} = \frac{A}{|w|}(\delta_{\alpha,\beta} - \frac{w_\alpha w_\beta}{|w|^2}) \tag{3.4.43}$$

and

$$A = \frac{1}{8\pi}\int_0^\infty d\rho\rho^3\hat{\phi}^2(\rho), \tag{3.4.44}$$

with $\hat{\phi}(|k|) = \int_{\mathbb{R}^3}\phi(r)e^{-ik\cdot r}$.

Proof. The first identity in (3.4.42) is due to the symmetry $F(r) = F(-r)$. Then we compute the left hand side of (3.4.42) taking the Fourier transform and passing in spherical coordinates. The result is

$$A\int_{S^2} d\hat{k}\delta(\hat{k}\cdot w)\hat{k}\otimes\hat{k} = a(w). \tag{3.4.45}$$

□ □

Finally by the use of the dominated convergence theorem we can establish

$$\lim_{\varepsilon\to 0}\int d\tau\mathcal{T}_1^{>}(\tau) = \int_0^t d\tau S(t-\tau)Q_L(S(\tau)f_0, S(\tau)f_0) \tag{3.4.46}$$

in $\mathcal{D}'$. To conclude the proof it remains to show that

$$\lim_{\varepsilon\to 0}\int_0^t d\tau\mathcal{T}_2^{>}(\tau) = 0. \tag{3.4.47}$$

We first evaluate

$$\begin{aligned}(h_\varepsilon - h)_{\alpha,\beta}(x_1, r, v_1, v_2, \tau, \varepsilon s) &= R_\alpha(x_1, v_1, \tau)\\ &\{[(\nabla_{v_1} - \nabla_{v_2})S(\tau-\varepsilon s)f_2^0](X_2(-\varepsilon s), V_2(-\varepsilon s)) - (\nabla_{v_1} - \nabla_{v_2})S(\tau)f_2^0(x_1, x_1, v_1, v_2)]\}_\beta\,.\end{aligned} \tag{3.4.48}$$

Note that

$$\nabla_v S(\tau)f(x,v) = S(\tau)(\nabla_v - \tau\nabla_x)f(x,v).$$

Omitting irrelevant variables we observe that

$$(h_\varepsilon - h)_{\alpha,\beta} = R_\alpha(\Phi_\beta(-\varepsilon s) - \Phi_\beta(0))$$

where $\Phi(\sigma) = [(\nabla_{v_1} - \nabla_{v_2})S(\tau+\sigma)f_2^0](X_2(\sigma), V_2(\sigma))$.
Hence

$$|h_\varepsilon - h| \le |R|\int_{-\varepsilon s}^0 d\sigma|\dot{\Phi}(\sigma)|.$$

It is easy to see that $\dot{\Phi}(\sigma)$ is a linear combination of various second derivatives of f_2^0, multiplied by $\dot{w}(\sigma) = \frac{2}{\sqrt{\varepsilon}}F\left(\frac{x_1(\sigma)-x_2(\sigma)}{\varepsilon}\right)$, plus two terms proportional to first derivatives with respect to x. All the derivatives are computed at the point $[X_2(\sigma) - (\tau+\sigma)V_2(\sigma), V_2(\sigma)]$. Hence, under the assumptions of Theorem 1, we obtain

$$|h_\varepsilon - h| \le C|R|\frac{1}{\sqrt{\varepsilon}}\int_0^{\varepsilon s} d\sigma\exp\{-b(|v_1(-\sigma)|^2 + |v_2(-\sigma)|^2)\}.$$

Since $|x_1(-\varepsilon s) - x_2(-\varepsilon s)| \geq \varepsilon$ if $s \geq 4/|w|$, the integral over ds in (3.4.24) can be estimated by

$$\begin{aligned}\left| \int_0^{\tau/\varepsilon} ds F_\beta \left(\frac{x_1(-\varepsilon s) - x_2(-\varepsilon s)}{\varepsilon} \right) (h_\varepsilon - h)_{\alpha,\beta} \right| &\leq \frac{C|R|}{\sqrt{\varepsilon}} \int_0^{4/|w|} ds \int_0^{\varepsilon s} d\sigma \psi(\sigma) \\ &\leq \frac{C|R|\sqrt{\varepsilon}}{|w|} \int_0^{4/|w|} d\sigma \psi(\varepsilon\sigma) ,\end{aligned} \tag{3.4.49}$$

where $\psi(\sigma) = \exp\{-b(|v_1(-\sigma)|^2 + |v_2(-\sigma)|^2)\}$.

Then by energy conservation

$$|v_1(t)|^2 + |v_2(t)|^2 + 2\sqrt{\varepsilon}\phi \left(\frac{x_1(t) - x_2(t)}{\varepsilon} \right) = const$$

and therefore

$$\psi(\varepsilon s) \leq A^2 \exp\{-b(|v_1|^2 + |v_2|^2 - 4\sqrt{\varepsilon}\|\phi\|_\infty)\}.$$

Hence, we obtain the following estimate of the integral (3.4.24):

$$|\mathcal{T}_2^>(\tau)| \leq C\sqrt{\varepsilon} \int dx_1 \int dr \int dV_2 \, |v_1 - v_2|^{-2} |R(x_1, v_1, \tau)| \, |F(r)| \, \exp\{-b(|v_1|^2 + |v_2|^2)\},$$

where $R(x_1, v_1, \tau)$ is given in (3.4.22). It is clear that $R(x_1, v_1, \tau) = 0$ if $|x_1| > R_1$, where R_1 depends only on τ. Therefore

$$|\mathcal{T}_2^>(\tau)| \leq C\sqrt{\varepsilon} \int dv_1 \int dv_2 |v_1 - v_2|^{-2} \exp\{-b(|v_1|^2 + |v_2|^2)\} = C_1\sqrt{\varepsilon}.$$

By Lemma 1 we also conclude that (3.4.12) holds and this completes the proof of Theorem 1. □ □

3.5 Propagation of chaos

In this section we extend the result obtained in Theorem 1 to the j-marginal distribution, showing the propagation of chaos (at first order in time). More precisely we have

Theorem 4. *Under hypotheses of Theorem 1, if* (3.3.11) *holds for all j, then*

$$\begin{aligned}\lim_{\varepsilon \to 0} \tilde{g}_j^N(t, x_1, v_1, \dots, x_j, v_j) = &\prod_{i=1}^{j} S(t) f_0(x_i, v_i) + \\ &+ \sum_{i=1}^{j} \prod_{\substack{k=1 \\ k \neq i}}^{j} S(t) f_0(x_k, v_k) \int_0^t d\tau S(t-\tau) Q_L(S(\tau)f_0, S(\tau)f_0)(x_i, v_i),\end{aligned} \tag{3.5.1}$$

$$\lim_{\varepsilon \to 0} \tilde{\gamma}_j^N(t, x_1, v_1, \dots, x_j, v_j) = 0 \tag{3.5.2}$$

in $\mathcal{D}'$.

Remark 2. *The reason why we call eq.n* (3.5.1) *propagation of chaos is that the r.h.s of* (3.5.1) *corresponds to the first order in time of* $\prod_{i=1}^{j} \lim_{\varepsilon\to 0} \tilde{g}_1^N(x_i, v_i, t)$.

Proof. Let $u \in \mathcal{D}(\mathbb{R}^{3j} \times \mathbb{R}^{3j})$ be a test function and let us consider

$$\begin{aligned}(u, \tilde{g}_j^N(t)) = (u, S(t)f_j^0) + \frac{N-j}{\sqrt{\varepsilon}} \int_0^t (u, S(t-\tau)C_{j+1}^{\varepsilon} S(\tau) f_{j+1}^0) d\tau \\ + \frac{N-j}{\sqrt{\varepsilon}} \int_0^t (u, S(t-\tau)C_{j+1}^{\varepsilon} \tilde{\gamma}_{j+1}^N(\tau)) d\tau.\end{aligned} \tag{3.5.3}$$

Of course the second term in (3.5.3) is of order $O(\sqrt{\varepsilon})$, hence we focus on the third term. Then, defining $R_i(X_j, V_j, \tau) := \nabla_{v_i} S(\tau - t) u(X_j, V_j)$, such a term is

$$\begin{aligned}T =& \frac{N-j}{\sqrt{\varepsilon}} \int_0^t (u, S(t-\tau) C_{j+1}^{\varepsilon}(\tau) \tilde{\gamma}_{j+1}^N) d\tau \\ =& -\frac{N-j}{\sqrt{\varepsilon}} \sum_{i=1}^{j} \int_0^t d\tau \int dX_{j+1} \int dV_{j+1}\, R_i(X_j, V_j, \tau) \cdot F(\frac{x_i - x_{j+1}}{\varepsilon}) \tilde{\gamma}_{j+1}^N(\tau) \\ =& -\frac{N-j}{\varepsilon} \sum_{i=1}^{j} \int_0^t d\tau \int dX_{j+1} \int dV_{j+1} \int_0^{\tau} ds R_i(X_j, V_j, \tau) \sum_{\substack{k,l=1 \\ k<l}}^{j+1} F\left(\frac{x_i - x_{j+1}}{\varepsilon}\right) \\ & \{F\left(\frac{x_k(-s) - x_l(-s)}{\varepsilon}\right) \cdot [(\nabla_{v_k} - \nabla_{v_l}) S(\tau - s) f_{j+1}^0](X_{j+1}(-s), V_{j+1}(-s))\}.\end{aligned} \tag{3.5.4}$$

We shall see that the leading term in the sum appearing in the r.h.s. of (3.5.4) is that with $k = i, l = j+1$, the other ones being vanishing. This is the content of the following

Lemma 5. *Let* $\varphi = \varphi(X_{j+1}, V_{j+1}, \tau, s) \geq 0$ *be a measurable function, compactly supported in* X_{j+1} *and such that*

$$\varphi \leq e^{-b|V_{j+1}|^2}.$$

Then, if $(k, l) \neq (i, j+1)$*, for all* i, k, l*, we have*

$$\frac{N-j}{\varepsilon} \int_0^{\tau} ds \int dX_{j+1} \int dV_{j+1} \varphi |F\left(\frac{x_i - x_{j+1}}{\varepsilon}\right)||F\left(\frac{x_k(-s) - x_l(-s)}{\varepsilon}\right)| \leq C^j \varepsilon\,. \tag{3.5.5}$$

Proof. We are integrating on the final coordinates $(X_{j+1}, V_{j+1}) = (X_{j+1}(0), V_{j+1}(0))$ of the flow $(X_{j+1}(\sigma), V_{j+1}(\sigma))$ defined for negative times $\sigma \in [-\tau, 0]$. We find convenient to reverse the velocities $V_{j+1} \to -V_{j+1}$ and look at positive times $s \in [0, \tau]$.

First of all we perform the usual change of variables $x_{j+1} = x_i - \varepsilon r$ and gain ε^3. Next we introduce the following partition of the phase space: setting $C_0 = \{(k,l), k < l | (k,l) \neq (i, j+1)\}$ we define

$$A_0(k,l) = \{(X_{j+1}, V_{j+1}) | \quad |x_k - x_l| < 2\varepsilon, (k,l) \in C_0\} \tag{3.5.6}$$

and

$$A_0 = \bigcup_{(k,l)\in C_0} A_0(k,l). \tag{3.5.7}$$

Furthermore, denoting by $s(k,l) \in [0,\tau]$ the first instant for which

$$|x_k(s) - x_l(s)| < \varepsilon \tag{3.5.8}$$

namely the pair of particles k and l starts to interact at time $s(k,l)$ (if they do not interact we set $s(k,l) = \tau$) we define:

$$A_{k,l} = \{(X_{j+1}, V_{j+1}) \notin A_0 | s(k,l) = \min_{(r,m)\in C_0} s(r,m) < \tau\}. \tag{3.5.9}$$

In other words if $(X_{j+1}, V_{j+1}) \in A_{k,l}$ the pair of particles $(k,l) \in C_0$ is the first interacting pair (excluded the pair $(i, j+1)$ which starts to interact at time 0) in the time interval $(0,\tau]$.

Note that we are interested to integrate over the set

$$A_0 \cup \bigcup_{(k,l)\in C_0} A_{k,l}. \tag{3.5.10}$$

In facts in the complement of the set (3.5.10), (3.5.5) vanishes because

$$|F\left(\frac{x_k(s) - x_l(s)}{\varepsilon}\right)| = 0.$$

To estimate the contribution due to $A_{k,l}$ we first assume that $k \neq i, l \neq j+1, i$.

Note that the motion of the pair of particles with indices (k,l) is free in $[0, s(k,l)]$. Then setting $x_k - x_l = y$ and $v_k - v_l = w$ we have

$$\inf_{s\in[0,\tau]} |y - ws| \leq \varepsilon. \tag{3.5.11}$$

The minimizing s is $s_0 = \frac{w\cdot y}{|w|^2}$ so that condition (3.5.11) yields

$$|y - w\frac{w\cdot y}{|w|^2}| \leq \varepsilon. \tag{3.5.12}$$

This means that the projection of y on the orthogonal plane to w is in the disk smaller than ε. Therefore

$$\frac{N-j}{\varepsilon}\int_{A(k,l)} dX_{j+1}dV_{j+1}\,\varphi\,|F\left(\frac{x_i - x_{j+1}}{\varepsilon}\right)||F\left(\frac{x_k(-s) - x_l(-s)}{\varepsilon}\right)| \leq C^j\varepsilon\,. \tag{3.5.13}$$

Now we consider the cases $k = i$, $l = i$ or $l = j+1$. For the sake of clearness we consider $k = i$, the other cases being completely analogous.

There are two possibilities: either $s(i,l) > \tilde{s}$, where $\tilde{s}$ is the last interaction time for the pair $(i, j+1)$, namely

$$|x_i(s) - x_{j+1}(s)| > \varepsilon$$

for $s > \tilde{s}$, or $s(i,l) \leq \tilde{s}$.

In the first case we can repeat the above argument setting $y = x_i(\tilde{s}) - x_l(\tilde{s})$ and $w = v_i(\tilde{s}) - v_l(\tilde{s})$.

In the second one observe that the center of mass $\bar{x} = \frac{x_i + x_{j+1}}{2}$ is moving freely with velocity $\bar{v} = \frac{v_i + v_{j+1}}{2}$ (because the pair $(i, j+1)$ is an isolated system at least up to a time $\bar{s} = s(i,l)$).

Condition

$$|x_i(\bar{s}) - x_l(\bar{s})| = \varepsilon$$

implies

$$|x_l(\bar{s}) - \bar{x}(\bar{s})| \le |x_i(\bar{s}) - x_l(\bar{s})| + |x_i(\bar{s}) - \bar{x}(\bar{s})| \le \frac{3}{2}\varepsilon \,. \tag{3.5.14}$$

Therefore we can integrate under the condition (3.5.14) to get

$$\frac{N-j}{\varepsilon} \int_{A(i,l)} dX_{j+1} dV_{j+1}\, \varphi \,|F\left(\frac{x_i - x_{j+1}}{\varepsilon}\right)||F\left(\frac{x_k(-s) - x_l(-s)}{\varepsilon}\right)| \le C^j \varepsilon \,. \tag{3.5.15}$$

Clearly we also have that

$$\frac{N-j}{\varepsilon} \int_{A_0} dX_{j+1} dV_{j+1}\, \varphi \,|F\left(\frac{x_i - x_{j+1}}{\varepsilon}\right)||F\left(\frac{x_k(-s) - x_l(-s)}{\varepsilon}\right)| \le C^j \varepsilon^2 \,. \tag{3.5.16}$$

Thus we conclude the proof. $\square$

Finally we handle the leading term. Setting

$$\begin{aligned} T_l = &- \frac{N-j}{\varepsilon} \sum_{i=1}^{j} \int_0^t d\tau \int dX_{j+1} \int dV_{j+1} \int_0^\tau ds R_i(X_j, V_j, \tau) F\left(\frac{x_i - x_{j+1}}{\varepsilon}\right) \\ &\left\{ F\left(\frac{x_i(-s) - x_{j+1}(-s)}{\varepsilon}\right) \cdot [(\nabla_{v_k} - \nabla_{v_l}) S(\tau - s) f^0_{j+1}](X_{j+1}(-s), V_{j+1}(-s)) \right\}, \end{aligned} \tag{3.5.17}$$

we have

Lemma 6. *The term with repeated indices is of order one. More precisely,*

$$\lim_{\varepsilon \to 0} T_l = \left(\sum_{i=1}^{j} \prod_{\substack{k=1 \\ k \ne i}}^{j} S(t) f_0(x_k, v_k) \int_0^t d\tau S(t-\tau) Q_L(S(\tau) f_0, S(\tau) f_0)(x_i, v_i), u \right). \tag{3.5.18}$$

Proof. At this point the proof is rather obvious and we only sketch it. We first reduce the integration domain in the definition of T_l for moderately large relative velocity, i.e. $|v_i - v_{j+1}| > a\varepsilon^{1/4}$, being the contribution of the complementary set negligible as we have seen in Section 3. Looking at

$$\begin{aligned} T_l^> = &- \frac{N-j}{\varepsilon} \sum_{i=1}^{j} \int_0^t d\tau \int dX_{j+1} \int_{|v_i - v_{j+1}| > a\varepsilon^{1/4}} dV_{j+1} \int_0^\tau ds R_i(X_j, V_j, \tau) \cdot \\ &F\left(\frac{x_i - x_{j+1}}{\varepsilon}\right) \left\{ F\left(\frac{x_i(-s) - x_{j+1}(-s)}{\varepsilon}\right) \cdot [(\nabla_{v_k} - \nabla_{v_l}) S(\tau - s) f^0_{j+1}](X_{j+1}, V_{j+1}) \right\}, \end{aligned} \tag{3.5.19}$$

we could apply the same argument as in Section 3 to get the result, if the motion of the pair of particles i and $j+1$ would be independent of the others. However we have seen in the proof of Lemma 5 that the contribution of the event in which the particle $k \neq i, j+1$ interacts with particle i or particle $j+1$ is indeed negligible. Hence (3.5.18) follows easily. $\square$

Finally, again by Lemma 1, we obtain (3.5.2). $\square$

Acknowledgements.
We thank the referee for having pointed out useful suggestions to improve the exposition of a previous version of this paper.
The support from Swedish Research Council (VR grant 621-2009-5751) for A.V.B. is gratefully acknowledged.

Bibliography

[1] A.A. Arsen'ev, O.E. Buryak, On a connection between the solution of the Boltzmann equation and the solution of the Landau–Fokker–Planck equation. (in Russian) Mat. Sb **181**, (1990) no. 2, 465–478.

[2] R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (John Wiley & Sons, New-York 1975).

[3] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, Some Considerations on the derivation of the nonlinear Quantum Boltzmann Equation. J. Stat. Phys. **116**, (2004) no. 114, 381–410.

[4] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, On The Weak–Coupling Limit for Bosons and Fermions. Math. Mod. Meth. Appl. Scie. **15**, (2005), no. 12, 1–33.

[5] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, From the N-body Schroedinger equation to the quantum Boltzmann equation: a term-by-term convergence result in the weak coupling regime. Comm. Math. Phys. **277** (2008), 1-44.

[6] A.V. Bobylev, Expansion of Boltzmann collision integral in Landau series (in Russian). Doklady AN SSSR **225** (1975), no. 3, 535–538; translation in Sov. Phys. Dokl. **20** (1976), 740–742.

[7] C. Cercignani, R. Illner, M. Pulvirenti, *The mathematical theory of dilute gases*, Applied Mathematical Sciences, Vol. 106, (Springer–Verlag, New York 1994).

[8] P. Degond, B. Lucquin–Desreux, The Fokker–Planck asymptotics of the Boltzmann collision operator in the Coulomb case. Math. Models Methods Appl. Sci. **2** (1992), no. 2, 167–182.

[9] L. Desvillettes, C. Villani, On the spatially homogeneous Landau equation for hard potentials. II. H-theorem and applications. Comm. Partial Differential Equations **25** (2000), 261–298.

[10] L. Desvillettes, C. Villani, On the spatially homogeneous Landau equation for hard potentials. I. Existence, uniqueness and smoothness. Comm. Partial Differential Equations **25** (2000), 179–259.

[11] D.Dürr, S. Goldstein, J. Lebowitz, Asymptotic motion of a classical particle in a random potential in two dimensions: Landau model. Comm. Math. Phys. **113** (1987), 209–230.

[12] L. Erdös, M. Salmhofer, H-T. Yau On the quantum Boltzmann equation. J. Stat. Phys. **116**, (2004), 367–380.

[13] T. Goudon, On Boltzmann equations and Fokker–Planck asymptotics: influence of grazing collisions. J. Statist. Phys. **89** (1997), no. 3-4, 751–776.

[14] Y. Guo, The Landau equation in a periodic box. Comm. Math. Phys. **231** (2002), no. 3, 391?434.

[15] N.T. Ho, L.J. Landau Fermi gas in a lattice in the van Hove limit. J. Stat. Phys **87**, (1997), 821–845.

[16] O. Lanford III *Time evolution of large classical systems.* Lecture Notes in Physics, **38**, E.J. Moser ed., (Springer, Verlag 1975), 1–111.

[17] L.D. Landau, Kinetic equation in the case of Coulomb interaction (in German). Phys. Zs. Sow. Union, **10** (1936), s. 154.

[18] E.M. Lifshitz, L.P. Pitaevskii, *Course of theoretical physics "Landau–Lifshit" vol. 10* (Pergamon Press, Oxford–Elmsford, N.Y. 1981).

[19] J. Lukkarinen, H. Spohn, Weakly nonlinear Schr?odinger equation with random initial data. Inventiones Mathematicae **183**, (2011), 79–188.

[20] N.N.Bogolyubov, *Problems of a Dynamical Theory in Statistical Physics* (State Techical Press, 1946, in Russian). English translation in Studies in Statistical Mechanics I, edited by J. de Boer and G. E. Uhlenbeck, part A, North-Holland, Amsterdam, 1962.

[21] M. Pulvirenti, The weak-coupling limit of large classical and quantum systems. *International Congress of Mathematicians* (Eur. Math. Soc., Zürich 2006).

[22] H. Spohn, *Large scale dynamics of interacting particles.* (Texts and monographs in physics, Springer, Heidelberg 1991).

[23] C.Villani, A review of mathematical topics in collisional kinetic theory. *Handbook of Mathematical Fluid Dynamics. Vol 1 ed. by S.Friedlander and D. Serre* (Elsevier, North Holland 2002), 71–307.

[24] C. Villani, On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. Arch. Rational Mech. Anal. **143**, (1998), no. 3, 273–307.

[25] E. Wigner, On the quantum correction for the thermodynamical equailibrium. Phys. Rev. **40**, (1932), 742–759.

[26] R. W. Zwanzig, Statistical mechanics of irreversibility. Lectures in theoretical physics, Vol. III (Interscience, New York 1961), 106–141.

Chapter 4

Grazing collision limit

In Sections 4.4–4.6 of the present Chapter we present [MPS].

4.1 Introduction

The Landau equation appears for the first time in 1936 in a paper by L.D. Landau. In [L-36] he derived the Landau collision operator starting from the Boltzmann collision integral, performing the so–called *grazing collision limit* (see Section 4.3 for details). The purpose of this Chapter is to adapt the grazing collision procedure by Landau -relying on physical considerations- to the toy model proposed by M. Kac in the 50s. Indeed, we have seen in Chapter 3 that a rigorous derivation of the Landau equation from a deterministic microscopic dynamics seems to be very difficult. We propose to start from a stochastic dynamics, as in the case of the Kac model for the homogeneous Boltzmann equation. In order to do that, we introduce a N-particle system which approaches, in the mean–field limit, the solutions of the Landau equation with Coulomb singularity. This model plays the same role as the Kac model for the homogeneous Boltzmann equation.

The plan of the Chapter is the following: in Section 4.2 we briefly review the model proposed by Kac in [K] for the homogeneous Boltzmann equation; an attempt to clarify the physical meaning of the grazing collision limit and the idea of the derivation of the Landau collision operator from the Boltzmann one is presented in Section 4.3; the remaining Sections are devoted to the study of the Kac's model for the Landau equation (we report exactly [MPS]). In particular the model we obtain by performing the grazing collision limit starting from the Kac model and the statement of the main result are presented in Section 4.4; Subsection 4.6.1 is addicted to some preliminary estimates used in the proof of Theorem 5 presented in Section 4.6, using compactness arguments inspired by [V-98].

4.2 Kac model

In 1954 M. Kac [K], in the attempt of clarifying some aspects of the transition from particle systems to the Boltzmann equation, introduced a toy model which has been successively investigated.

He started from the Boltzmann equation for hard spheres of diameter ε, performing elastic collisions; in this simple case, the Boltzmann equation can be written as follows:

$$\begin{aligned}\frac{\partial f}{\partial t}(t,x,v)+v\cdot\nabla_x f(t,x,v) = \frac{\varepsilon^2}{2}\int dv_1 \int dn |(v_1-v)\cdot\nu| \\ \{f(t,x,v+(v_1-v)\cdot\nu\,\nu)f(t,x,v_1-(v_1-v)\cdot\nu\,\nu)-f(t,x,v)f(t,x,v_1)\}\ ,\end{aligned} \tag{4.2.1}$$

where f denotes the probability density of a gas in a region Λ and ν is a unit vector. Since there are no external forces, Kac assumed the probability density to be of the following form:

$$f(t,x,v) = \frac{N}{|\Lambda|} f(t,v)\ , \tag{4.2.2}$$

where N is the total number of particles of the gas confined in the region Λ, whose volume is denoted by $|\Lambda|$; this means that the probability density is factorized in a term which involves the spatial properties (in fact the ratio between the number of particles and the volume of the region in which they are contained is essentially the definition of the spatial density) and a function which represents the velocity density. It is easy to see by substituting (4.2.2) in (4.2.1) that $f(t,v)$ is a solution to the reduced homogenous Boltzmann equation[1]

$$\begin{aligned}\frac{\partial f}{\partial t}(t,v) = & \frac{\varepsilon^2}{2}\frac{N}{|\Lambda|}\int dv_1 \int d\nu |(v_1-v)\cdot\nu| \\ & \{f(t,v+(v_1-v)\cdot\nu\,\nu)f(t,v_1-(v_1-v)\cdot\nu\,\nu)-f(t,v)f(t,v_1)\}\ ,\end{aligned} \tag{4.2.3}$$

describing the time evolution of the velocity density with uniform spatial density.

Kac's idea was to look at the reduced equation (4.2.3) to give a new probabilistic formulation of the problem.

Roughly speaking the Kac's model consists in a system of N particles with associated velocities $\mathbf{v}_N = (v_1,\dots,v_N)\in\mathbb{R}^{3N}$, whose dynamics is the following stochastic process: at a random time chosen accordingly to a Poisson process, pick a pair of particles, say i and j, and perform the transition

$$v_i, v_j \longmapsto v_i', v_j'$$

preserving total momentum and energy. More precisely, in the case of hard spheres, v_i' and v_j' are given by the usual laws of elastic collisions:

$$\begin{cases} v_i' = v_i + (v_j - v_i)\cdot\nu\,\nu\ , \\ v_j' = v_j - (v_j-v_i)\cdot\nu\,\nu\ . \end{cases} \tag{4.2.4}$$

[1]We observe that if the system we want to consider is not the hard–sphere one, we have to replace the factor $\frac{\varepsilon^2}{2}|(v_1 - v\cdot\nu)|$ by a function depending on the interaction (see the discussion in Chapter 2 for details).

The probability of such a "collision" event is assumed to be a function of the modulus of the relative velocities of the two particles involved in the collision and of the angle between their relative velocity and the unit vector ν. In the case of hard spheres, this probability has the following structure:

$$K_{i,j} = \frac{\varepsilon^2}{2} \frac{|(v_j - v_i) \cdot \nu| - (v_j - v_i) \cdot \nu}{|\Lambda|} \, d\nu \, dt \, . \tag{4.2.5}$$

It is interesting to observe that Kac chose the above expression for the transition probability because it is equivalent to the "Stosszahlansatz" originally formulated by Boltzmann in 1872. Indeed, in [K] is stated that the Boltzmann's hypothesis of chaos is the following: let us consider two particles, say particle i and particle j, whose velocities are v_i and v_j, then the number of collisions in the interval time dt between particle i and particle j which take place when the line joining the centers of the two particles are in the direction ν is

$$f(v_i)f(v_j)\frac{\varepsilon^2}{2} \{|(v_j - v_i) \cdot \nu| - (v_j - v_i) \cdot \nu\} \, d\nu \, dt \, , \tag{4.2.6}$$

where f is the average number of particles. If such a collision takes place, the velocity vector changes according to the usual scattering laws, preserving momentum and energy in each collision.

The master equation for this model is just the Kolmogorov equation associated to the Markov process we are considering. Indeed let $f^N(0, \mathbf{v}_N)$ be the symmetric (in the exchange of variables) N-joint probability density at time zero; its evolution is given by

$$\frac{\partial f^N}{\partial t}(t, \mathbf{v}_N) = \sum_{1 \le i < j \le N} \int d\nu \, K_{i,j}(\nu) \{f^N(t, v_1, \ldots, v_i', \ldots, v_j', \ldots, v_N) - f^N(t, v_1, \ldots, v_N)\} \, . \tag{4.2.7}$$

Further simplifications of the model [2] lead to the well–known Kac model.

More precisely, if $f^N = f^N(\mathbf{v}_N, t)$ is a symmetric probability distribution describing a statistical state of the system, the time evolution is given by the following master equation

$$\partial_t f^N = \mathcal{L}_N f^N \tag{4.2.8}$$

where

$$\begin{aligned} \mathcal{L}_N f^N = \frac{1}{2N} \sum_{i \neq j} \int & dv_i' \, dv_j' \, K(v_i, v_j | v_i', v_j') \delta(v_i + v_j - v_i' - v_j') \delta(v_i^2 + v_j^2 - v_i'^2 - v_j'^2) \\ & \{f^N(v_1, \ldots, v_i', \ldots, v_j', \ldots, v_N) - f^N(v_1, \ldots, v_N)\}, \end{aligned} \tag{4.2.9}$$

and K is a suitable kernel.

Introducing the exchanged momentum $p = v_i' - v_i = v_j - v_j'$ in the collision process and assuming that

$$K(v_i, v_j | v_i', v_j') = w(p) \tag{4.2.10}$$

[2]The simplifications made by Kac are the loss of the conservation of momentum and the simplification of the expression of the kernel.

for some smooth and radially symmetric w, we readily arrive to

$$\mathcal{L}_N f^N = \frac{1}{2N} \sum_{i \neq j} \int dp\, w(p) \delta\left(p^2 - p \cdot (v_i - v_j)\right) \tag{4.2.11}$$
$$\{f^N(v_1, \ldots, v_i + p, \ldots, v_j - p, \ldots, v_N) - f^N(v_1, \ldots, v_N)\}.$$

In [K] it was shown that the first marginal of f^N converges, in the limit $N \to \infty$, to the solution to the (homogeneous) Boltzmann equation if the initial datum is chaotic, i. e. if $f^N(0) = f_0^{\otimes N}$ for some probability distribution f_0. Moreover, the j-particle marginal converges to the j-fold product of such solution, i.e. propagation of chaos holds (see (4.5.15) below).[3]

4.3 Grazing collision limit: from Boltzmann to Landau

If we are interested in the study of the probability density of a rarefied gas, the Boltzmann equation is the right model to look at. On the other hand, if the gas is dense, the interactions among particles are of Coulomb type so that singularities appear and the integrals become divergent when distances among particles are large, so that the Boltzmann collision operator makes no sense. To study these phenomena, in 1936 Landau [TH], starting from the Boltzmann collision operator, derived a new kinetic equation for the time evolution of the probability density of a dense charged plasma, exploiting the fact that, in this physical context, only the grazing collisions ($p \approx 0$) are relevant. In this Section we shall reproduce Landau's approach and point out some relevant features associated to the Landau equation.
First of all, Landau started from physical observations, pointing out that the right forces acting in a plasma are of Coulomb nature, so that a variation of the particle motion is possible at large distances too. If we use the Boltzmann collision operator, Coulomb forces would produce divergences in the integrals for large distances among particles that are colliding. This fact points out a first key feature in the description of the evolution of the density function of a plasma: collisions such that distances among the colliding particles are large are essential.
We notice that the relevance of large distances is linked to the fundamental role played by little variations of the momentum; indeed at large distances the particles change their trajectories if the variation of momentum is small[4]; more precisely, particles at large distances could only

[3]**Theorem** (Kac - 1954)*Let $f^N(0)$ be a sequence of probability density functions having the "Boltzmann property"*

$$\lim_{N \to \infty} f_j^N(\mathbf{v}_j,\, 0) = \prod_{k=1}^{j} \lim_{N \to \infty} f_1^N(v_k,\, 0)\ .$$

Then $f^N(t)$, solutions to (4.5.1)*, also have the "Boltzmann property":*

$$\lim_{N \to \infty} f_j^N(\mathbf{v}_j,\, t) = \prod_{k=1}^{j} \lim_{N \to \infty} f_1^N(v_k,\, t)\ .$$

[4]The smallness in the variation of momentum can be considered as a diffusion in velocity.

scatter through small angles and small change of velocity, so that collisions in which velocities are slightly changed are important.

Landau's approach (see also [P]) is the following: let us consider the Boltzmann collision integral in the equivalent form

$$Q(f,f) = \int dv_1 \int dv' \int dv_1' K(v, v_1|v', v_1')\{f(v')f(v_1') - f(v)f(v_1)\}\,, \tag{4.3.1}$$

where v' and v_1' are obtained according to (4.2.4) and $K(v, v_1|v', v_1')$ is a function of velocity variables before and after the collision and it takes into account the conservation of momentum and energy; more precisely

$$K(v, v_1|v', v_1') = w(p)\delta(v + v_1 - v' - v_1')\delta(v^2 + v_1^2 - (v')^2 - (v_1')^2)\,, \tag{4.3.2}$$

where p is the transferred momentum, i.e. $p = v' - v = v_1 - v_1'$.

In order to express the smallness of the transferred momentum, we consider a small scale parameter $\varepsilon > 0$ and we rescale the function w in such a way that the grazing collisions are relevant in the limit of ε small:

$$w(p) \longrightarrow \frac{1}{\varepsilon^3} w\left(\frac{p}{\varepsilon}\right)\,.$$

To take into account the high density of the plasma, we also rescale the mean–free path:

$$\frac{1}{\lambda} \longrightarrow \frac{1}{\varepsilon\lambda}\,,$$

and for simplicity we take $\lambda = 1$.

The rescaled collision integral is

$$Q_\varepsilon(f,f) = \frac{1}{\varepsilon^4} \int dv_1 \int dp\, w\left(\frac{p}{\varepsilon}\right) \delta(p^2 + (v - v_1)\cdot p)\{f(v')f(v_1') - f(v)f(v_1)\}\,. \tag{4.3.3}$$

Performing the change of variable $p/\varepsilon = \tilde{p}$, the expression of the collision integral becomes

$$\begin{aligned} Q_\varepsilon(f,f) =& \frac{1}{\varepsilon^4} \int dv_1 \int \varepsilon\, d\tilde{p}\, w(\tilde{p})\delta(\tilde{p}^2\varepsilon^2 + (v - v_1)\cdot \tilde{p}\varepsilon)\{f(v')f(v_1') - f(v)f(v_1)\} \\ =& \frac{1}{\varepsilon^2} \int dv_1 \int d\tilde{p}\, w(\tilde{p})\delta(\tilde{p}^2\varepsilon + (v - v_1)\cdot \tilde{p})\{f(v')f(v_1') - f(v)f(v_1)\}\,. \end{aligned}$$

We observe that for every $x,\ y\ \in\ \mathbb{R}^3$

$$\delta(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} ds\ e^{isx\cdot y}\,. \tag{4.3.4}$$

Hence we use (4.3.4) to rewrite (4.3.3) as

$$Q_\varepsilon(f,f) = \frac{1}{\varepsilon^2} \int dv_1 \int d\tilde{p}\, w(\tilde{p})\, \frac{1}{2\pi}\int_{-\infty}^{+\infty} ds\ e^{is(\varepsilon\tilde{p}^2 + (v - v_1)\cdot\tilde{p})}\{f(v')f(v_1') - f(v)f(v_1)\}\,;$$

by (4.2.4), we notice that $v + \varepsilon\tilde{p} = v'$ and $v_1 - \varepsilon\tilde{p} = v_1'$, so that (4.3.3) is equal to

$$\frac{1}{\varepsilon^2} \int dv_1 \int d\tilde{p}\, w(\tilde{p})\, \frac{1}{2\pi}\int_{-\infty}^{+\infty} ds\ e^{is(\varepsilon\tilde{p}^2 + (v - v_1)\cdot\tilde{p})}\{f(v + \varepsilon\tilde{p})f(v_1 - \varepsilon\tilde{p}) - f(v)f(v_1)\}\,;$$

and by standard manipulations we obtain

$$\frac{1}{\varepsilon^2}\int dv_1\int d\tilde{p}\,w(\tilde{p})\int_0^1 d\lambda\,\frac{1}{2\pi}\int_{-\infty}^{+\infty} ds\; e^{is(\varepsilon\tilde{p}^2+(v-v_1)\cdot\tilde{p})}\varepsilon\tilde{p}\cdot(\nabla_v-\nabla_{v_1})f(v+\varepsilon\tilde{p})f(v_1-\varepsilon\tilde{p})\ .$$

We underline that we assumed $w(\cdot)$ to be an even function, depending only on $|p|$.
To investigate the behavior of the rescaled collision integral in the limit $\varepsilon\to 0$, we consider the weak formulation of the problem. Let φ be a test function and let us denote by $(\cdot\ ,\ \cdot)$ the inner product in L^2, as usual; then

$$\begin{aligned}(Q_\varepsilon(f,f),\varphi) = \frac{1}{2\pi\varepsilon}\int dv\int dv_1\int d\tilde{p}\,w(\tilde{p})\int_0^1 d\lambda\int_{-\infty}^{+\infty} ds e^{is(\tilde{p}^2(\varepsilon-2\varepsilon\lambda)+(v-v_1)\cdot\tilde{p})}\varphi(v-\varepsilon\lambda\tilde{p})\times\\ \times\,\tilde{p}\cdot(\nabla_v-\nabla_{v_1})f(v)f(v_1)\ .\end{aligned}$$

We expand the above expression in power of ε:

$$\begin{aligned}(Q_\varepsilon(f,f),\varphi) =&\frac{1}{2\pi\varepsilon}\int dv\int dv_1\int d\tilde{p}\,w(\tilde{p})\int_0^1 d\lambda\int_{-\infty}^{+\infty} ds e^{is(v-v_1)\cdot\tilde{p})}\{\varphi(v)+\varepsilon\tilde{p}\cdot\nabla_v\varphi(v)\}\times\\ &\times\,\tilde{p}\cdot(\nabla_v-\nabla_{v_1})f(v)f(v_1)+\\ &+\frac{1}{2\pi}\int dv\int dv_1\int d\tilde{p}\,w(\tilde{p})\int_{-\infty}^{+\infty} e^{is(v-v_1)\tilde{p}}\varphi(v)is\tilde{p}^2\int_0^1 d\lambda(1-2\lambda)\times\\ &\times\,\tilde{p}\cdot(\nabla_v-\nabla_{v_1})f(v)f(v_1)+\mathcal{O}(\varepsilon)=\\ =&\mathcal{I}_1+\mathcal{I}_2+\mathcal{O}(\varepsilon)\ .\end{aligned}$$

Using the fact the $w(\cdot)$ is an even function, $\mathcal{I}_1$ vanishes; moreover the imaginary part of $\mathcal{I}_1$ is also zero because the integral in $d\lambda$ vanishes; then the collision integral is equal to the real part of $\mathcal{I}_2$ plus $\mathcal{O}(\varepsilon)$:

$$\begin{aligned}(Q_\varepsilon(f,f),\varphi) =&\frac{1}{2\pi}\int dv\int dv_1\int d\tilde{p}\,w(\tilde{p})\int_{-\infty}^{\infty} ds e^{is(v-v_1)\cdot\tilde{p}}\tilde{p}\cdot\nabla_v\varphi(v)\times\\ &\times\,\tilde{p}\cdot(\nabla_v-\nabla_{v_1})f(v)f(v_1)+\mathcal{O}(\varepsilon)\ .\end{aligned}$$

Formally, we yield to

$$\lim_{\varepsilon\to 0}(Q_\varepsilon(f,f),\varphi) = (Q_L(f,f),\varphi)\ , \tag{4.3.5}$$

where Q_L is the Landau collision operator[5]:

$$Q_L(f,f)(t,v) = \int dv_1\nabla_v\cdot a(v-v_1)(\nabla_v-\nabla_{v_1})f(t,v)f(t,v_1)\ , \tag{4.3.6}$$

with $a(\cdot)$ the matrix

$$a(v-v_1) = \int d\tilde{p}\,w(\tilde{p})\delta((v-v_1)\cdot\tilde{p})\tilde{p}_i\tilde{p}_j\ . \tag{4.3.7}$$

[5] We observe that the Landau collision operator is expressed in divergence form, coherently with the footnote number 4 of the present Chapter.

We observe that (4.3.7) can be written in polar coordinates as

$$
\begin{aligned}
a_{i\,j}(v-v_1) =& \frac{1}{|v-v_1|}\int d\tilde{p}\,|\tilde{p}|\,w(\tilde{p})\delta\left(\frac{(v-v_1)}{|v-v_1|}\cdot\frac{\tilde{p}}{|\tilde{p}|}\right)\frac{\tilde{p}_i}{|\tilde{p}|}\frac{\tilde{p}_j}{|\tilde{p}|} = \\
=& \frac{A}{|v-v_1|}\int d\hat{p}\delta\left(\frac{(v-v_1)}{|v-v_1|}\cdot\hat{p}\right)\hat{p}_i\hat{p}_j\ ,
\end{aligned}
$$

where $\hat{p} = \tilde{p}/|\tilde{p}|$ and

$$
A = \int_0^{+\infty} dr\, r^3 w(r)\ . \tag{4.3.8}
$$

This implies that, for every $y \in \mathbb{R}^3$, the matrix $a(y)$ is composed by elements

$$
a_{i\,j}(y) = \frac{A}{|y|}\left(\delta_{i\,j} - \frac{y_i y_j}{|y|^2}\right)\ , \qquad \forall\ i,\ j\ = 1,\ 2,\ 3\ .
$$

We notice that the nature of the interaction expressed by the function $w(\cdot)$ is lost in the limit, appearing only in the constant A, as it is clear by (4.3.8).
From now on the main purpose of the present Chapter is to introduce the analogous of the Kac model for the Landau equation with Coulomb interaction (4.3.6).

In the following Sections we report the paper [MPS].

4.4 A Kac model for the Landau equation

Abstract. We introduce a N-particle system which approaches, in the mean-field limit, the solutions of the Landau equation with Coulomb singularity. This model plays the same role as the Kac's model for the homogeneous Boltzmann equation. We use compactness arguments following [11].

4.5 Introduction

In 1954 M. Kac [6], in the attempt of clarifying some aspects of the transition from particle systems to the Boltzmann equation, introduced a toy model which has been successively investigated. See for instance [9] and references quoted therein.

Roughly speaking the Kac's model consists in a N-particle system. The particles have no position but only velocities denoted by $V_N = (v_1, \dots, v_N) \in \mathbb{R}^{3N}$. The dynamics is the following stochastic process. At a random time, pick a pair of particles, say i and j, and perform the transition

$$
v_i, v_j \to v_i', v_j'
$$

preserving total momentum and energy.

More precisely, if $W^N = W^N(V_N, t)$ is a symmetric probability distribution describing a statistical state of the system, the time evolution is given by the following master equation

$$\partial_t W^N = \mathcal{L}_N W^N \tag{4.5.1}$$

where

$$\begin{aligned}\mathcal{L}_N W^N = \frac{1}{2N} \sum_{i \neq j} \int dv_i'\, dv_j'\, K(v_i, v_j | v_i', v_j') \delta(v_i + v_j - v_i' - v_j') \delta(v_i^2 + v_j^2 - v_i'^2 - v_j'^2) \\ \{W^N(v_1, \dots, v_i', \dots, v_j', \dots, v_N) - W^N(v_1, \dots, v_N)\},\end{aligned} \tag{4.5.2}$$

and K is a suitable kernel.

Introducing the exchanged momentum $p = v_i' - v_i = v_j - v_j'$ in the collision process and assuming that

$$K(v_i, v_j | v_i', v_j') = w(p) \tag{4.5.3}$$

for some smooth and radially symmetric w, we readily arrive to

$$\begin{aligned}\mathcal{L}_N W^N = \frac{1}{2N} \sum_{i \neq j} \int dp\, w(p) \delta\left(p^2 - p \cdot (v_i - v_j)\right) \\ \{W^N(v_1, \dots, v_i + p, \dots, v_j - p, \dots, v_N) - W^N(v_1, \dots, v_N)\}.\end{aligned} \tag{4.5.4}$$

In [6] it was shown that the first marginal of W^N converges, in the limit $N \to \infty$, to the solution to the (homogeneous) Boltzmann equation if the initial datum is chaotic, i. e. if $W^N(0) = f_0^{\otimes N}$ for some probability distribution f_0. Moreover, the j-particle marginal converge to the j-fold product of such solution, i.e., propagation of chaos holds (see (4.5.15) below).

The main purpose of the present paper is to introduce an analogous model for the Landau equation with Coulomb interaction. A straightforward way to derive this model is to perform the so-called grazing collision limit on eq.n (4.5.1) as we shall do in a moment. In fact in 1936 Landau [8], starting from the Boltzmann collision operator, derived a new kinetic equation for the time evolution of a dense charged plasma, exploiting the fact that, in this physical context, only the grazing collisions ($p \approx 0$) are relevant. According to such a prescription, we introduce $\varepsilon > 0$ a small parameter and scale the kernel of $\mathcal{L}_N$ in eq.n (4.5.4) as

$$w(p) \to \frac{1}{\varepsilon^3} w\left(\frac{p}{\varepsilon}\right)$$

so that

$$\begin{aligned}\mathcal{L}_N^\varepsilon W^N = \frac{1}{2N\varepsilon^4} \sum_{i \neq j} \int dp\, w\left(\frac{p}{\varepsilon}\right) \delta\left(p^2 - p \cdot (v_i - v_j)\right) \\ \{W^N(v_1, \dots, v_i + p, \dots, v_j - p, \dots, v_N) - W^N(v_1, \dots, v_N)\}.\end{aligned} \tag{4.5.5}$$

Note that we inserted another factor $1/\varepsilon$ in front of the collision operator, to take into account the large density of the plasma.

Now, for fixed N, we perform the limit $\varepsilon \to 0$. By a straightforward formal computation (change of variables and Taylor expansion), we readily detect the limiting generator which is the following diffusion operator:

$$\tilde{L}^N = \mathrm{div}_{V_N}(B \cdot \nabla_{V_N}). \tag{4.5.6}$$

Here

$$B : \mathbb{R}^{3N} \to \mathbb{R}^{3N \times 3N}$$

is a matrix defined in the following way. For $V_N = (v_1, \dots, v_N) \in \mathbb{R}^{3N}$,

$$\begin{cases} B_{i,j}(V_N) = -\dfrac{a(v_i - v_j)}{N} & \text{if } \ i \neq j, \\ B_{i,i}(V_N) = \dfrac{1}{N} \displaystyle\sum_j a(v_i - v_j), \end{cases}$$

where the 3×3 matrix a is given by

$$a(w) = \frac{1}{|w|}(\mathbb{I} - \hat{w} \otimes \hat{w}) = \frac{1}{|w|} P(w), \quad w \in \mathbb{R}^3, \quad \text{and } \hat{w} = \frac{w}{|w|}, \tag{4.5.7}$$

with $P(w)$ the orthogonal projection on the plane orthogonal to w.

Unfortunately the elliptic operator $\tilde{L}^N$ has two main disadvantages. First it is not uniformly elliptic (see Lemma 7 below), second it is not smooth due to the divergence for $|v_i - v_j| \approx 0$.

As a matter of fact, since we want a handier N-particle model to start with, we slightly modify $\tilde{L}^N$ to obtain a smooth and non-degenerate operator. More precisely, we define

$$L^N = \mathrm{div}_{V_N}(B^N \nabla_{V_N}) \tag{4.5.8}$$

where B^N is obtained by making the matrix B smooth and bounded from below:

$$\begin{cases} B^N_{i,j}(V_N) = -\dfrac{a^N(v_i - v_j)}{N} & \text{if } \ i \neq j, \\ B^N_{i,i}(V_N) = \dfrac{1}{N} \displaystyle\sum_j a^N(v_i - v_j) + \dfrac{1}{N}. \end{cases} \tag{4.5.9}$$

Here the 3×3 matrix a^N is obtained by replacing $\frac{1}{|w|}$ by $\bar{\chi}_{\frac{1}{N}}(|w|)\frac{1}{|w|}$ in (4.5.7), defining

$$\chi_{\frac{1}{N}} \in C^\infty(\mathbb{R}^+), \quad \chi_{\frac{1}{N}}(r) = 1 \quad \text{if} \quad r < \frac{1}{N}, \quad \chi_{\frac{1}{N}}(r) = 0 \quad \text{if} \quad r > \frac{2}{N}, \tag{4.5.10}$$

and $\bar{\chi}_N = (1 - \chi_N)$. Now the evolution equation assumes the form

$$\partial_t W^N = \mathrm{div}_{V_N}(B^N \nabla_{V_N} W^N) \tag{4.5.11}$$

and the well-known theory of linear parabolic equations assures the existence of a unique classical solution for L^1 initial data.

To simplify the notations we define

$$\frac{1}{|w|_N} := \bar{\chi}_{\frac{1}{N}}(|w|)\frac{1}{|w|}$$

so that $a^N(w) = \frac{1}{|w|_N}P(w)$.

In the limit $N \to \infty$, the number of variables in the definition of W^N diverges, hence we will actually prefer to look at the asymptotic behavior of the *marginal distributions*

$$f_j^N(v_1, \ldots, v_j, t) = \int dv_{j+1} \ldots dv_N \, W^N(v_1, \ldots, v_N, t), \quad j = 1, \ldots, N.$$

Note that $f_N^N = W^N$ and the j-th marginal distribution is a function of j variables. Moreover, using (4.5.11) we can express the evolution of each f_j^N in terms of f_{j+1}^N. Straightforward computations lead to the following system of equations, called the N-particle *hierarchy*

$$\partial_t f_j^N = L_j^N f_j^N + \frac{N-j}{N} C_{j+1}^N f_{j+1}^N, \quad j = 1, \ldots, N-1 \tag{4.5.12}$$

where L_j^N and C_{j+1}^N are operators defined by:

$$\begin{aligned} L_j^N f_j^N &= \frac{1}{N} \sum_{\substack{k \neq l \\ k,l=1}}^{j} \nabla_{v_k} \cdot \left[a_{k,l}^N \cdot (\nabla_{v_k} f_j^N - \nabla_{v_l} f_j^N) \right] + \frac{1}{N} \sum_{k=1}^{j} \Delta_{v_k} f_j^N, \\ C_{j+1}^N f_{j+1}^N &= \sum_{k=1}^{j} \nabla_{v_k} \cdot \int dv_{j+1} \, a_{k,j+1}^N \cdot \left(\nabla_{v_k} f_{j+1}^N - \nabla_{v_{j+1}} f_{j+1}^N \right). \end{aligned} \tag{4.5.13}$$

In particular we have $L_N^N = L^N$.

Since $C_j = O(1)$, while $L_j^N f_j^N = O(\frac{j}{N})$, the formal limit of (4.5.12) as $N \to \infty$ yields an infinite system of equations called *Landau hierarchy*

$$\partial_t f_j = C_{j+1} f_{j+1}, \quad j = 1, \ldots, +\infty, \tag{4.5.14}$$

where the operators C_{j+1} write

$$C_{j+1} g = \sum_{k=1}^{j} \nabla_{v_k} \cdot \int dv_{j+1} \, a_{k,j+1} \cdot \left(\nabla_{v_k} g - \nabla_{v_{j+1}} g \right).$$

Due to the structure of the collision operator C_{j+1}, we realize that special solutions to eq.n (4.5.14) are given by factorized states

$$f_j(v_1 \ldots v_j, t) = \prod_{i=1}^{j} f(v_i, t) = f(t)^{\otimes j} \tag{4.5.15}$$

where the one particle distribution $f(t)$ solves the Landau equation

$$\partial_t f = Q(f, f), \tag{4.5.16}$$

with

$$Q(f,f)(v) = \int_{\mathbb{R}^3} dw\, a(v-w) \cdot (f(w)\nabla f(v) - f(v)\nabla f(w)). \tag{4.5.17}$$

It should be mentioned that, conversely, if f is a solution to eq.n (4.5.16), then the products $f_j = f^{\otimes j}$ solve the hierarchy (4.5.14).

Following the general paradigm of the kinetic theory, we expect that propagation of chaos holds, namely that (4.5.15) holds for all time provided that the initial state is chaotic, i.e. (4.5.15) is initially verified.

Actually, we are not able to show propagation of chaos. We will be able to prove only the (weak) convergence $f_j^N(t) \to f_j(t)$ (for suitable subsequences), being $f_j(t)$ a weak solution of the Landau hierarchy (4.5.14), without knowing whether $f_j(t)$ factorizes even though it does at time zero. The reason is that we have a poor control on the limiting hierarchy as well as on the Landau equation (4.5.16). In fact, we will obtain a solution to eq.n (4.5.14) by adapting, to the present N-particle context, a strategy, based on compactness arguments, introduced by C. Villani [11] for the Landau equation. As a matter of fact we do not have uniqueness, which is a necessary condition to get propagation of chaos. Indeed, assume that $f(t)$ and $g(t)$ are two weak solutions to eq.n (4.5.16), with the same initial datum f_0. It follows that

$$f_j(t) = \lambda f(t)^{\otimes j} + (1-\lambda) g(t)^{\otimes j}, \quad \lambda \in (0,1)$$

solves the Landau hierarchy with the chaotic initial datum $f_0^{\otimes j}$, but does not factorize.

Before stating our main result, we make some assumptions on the initial a.c. measures $W^N(0)$:

1. $W^N(0) \geq 0$;
2. $W^N(0)$ is symmetric in the variables $v_1, \dots, v_N$;
3. The following uniform bounds hold

$$\begin{aligned}
&\int dV_N\, W^N(0) = 1, \\
&\frac{1}{N}\int dV_N\, W^N(0)\log(W^N(0)) \leq C, \\
&\frac{1}{N}\int dV_N\, W^N(0)|V_N|^2 \leq C.
\end{aligned}$$

These properties still hold true at positive times. Actually

$$\int dV_N\, W^N(t)|V_N|^2 = \int dV_N\, W^N(0)|V_N|^2 + \frac{C}{N}t$$

expresses the energy dissipation and follows easily by an integration by parts in eq.n (4.5.11). Moreover

$$\int dV_N\, W^N(t)\log(W^N(t)) \leq \int dV_N\, W^N(0)\log(W^N(0))$$

expresses the entropy dissipation and will be discussed in the next section.

We now explain what is the sense we give to eq.n (4.5.14). The main difficulty related to the Landau equation is due to the divergence of the matrix $a(w)$ when $|w|$ is small. Indeed if f_{j+1} (some weak limit of f^N_{j+1}) is only in $L^1(\mathbb{R}^{3(j+1)})$, the integral

$$\int f_{j+1}(v_1, \dots, v_{j+1}) \frac{1}{|v_i - v_{j+1}|}$$

makes no sense; therefore $C_{j+1} f_{j+1}$ is not defined in general. Thus, as we did before in (4.5.10) to regularize the operator $\tilde{L}^N$, we introduce a small parameter $\delta > 0$ and the cut-off function $\chi_\delta \geq 0$, not increasing and such that

$$\chi_\delta \in C^\infty(\mathbb{R}^+), \quad \chi_\delta(r) = 1 \quad \text{if} \quad r < \delta, \quad \chi_\delta(r) = 0 \quad \text{if} \quad r > 2\delta. \tag{4.5.18}$$

Then we define C^δ_{j+1} replacing $a(w)$ in definition (4.5.13) by $a(w)(1-\chi_\delta(|w|)$, thus removing the singularity. Clearly, if $\varphi \in C^2_c$ then $\int \varphi C^\delta_{j+1} f_{j+1}$ makes sense for any $f_{j+1} \in \mathcal{M}(j+1)$, where $\mathcal{M}(k), k \geq 0$, denotes the space of probability measures on $\mathbb{R}^{3k}$ equipped with the topology given by the weak convergence of probability measures.

Our result can be stated as follows

Theorem 5. *There exists a subsequence $N_k \to \infty$ such that, for all j, there exists $f_j \in L^\infty([0,T]; L^1) \cap C^0([0,T]; \mathcal{M}(j))$, with finite mass, energy and entropy, such that*

$$f^{N_k}_j \to f_j \quad \textit{when} \quad k \to \infty,$$

where the convergence holds in the sense of weak convergence of probability measures. For any $t > 0$ and for any test function $\varphi \in C^2_c(\mathbb{R}^{3j})$, the limit

$$\lim_{\delta \to 0} \int_0^t ds \int dv_1 \dots dv_j \, \varphi(v_1, \dots, v_j) C^\delta_{j+1} f_{j+1}(v_1, \dots, v_j, s), \quad j = 1, \dots, +\infty$$

exists, and we have

$$\int \varphi f_j(t) - \int \varphi f_j(0) = \int_0^t ds \int \varphi C_{j+1} f_{j+1}(s), \quad j = 1, \dots, \infty.$$

Remark 3. *Following [11], as we shall see in the proof of Theorem 5 we have more regularity on f_j (see (4.6.15)). This allows us to give a direct sense to C_{j+1} without using a cut-off function.*

We conclude this section with some additional remarks.

Another kind of Landau equations can also be considered replacing the matrix a by

$$a_\alpha(w) = \frac{1}{|w|^\alpha}(\mathbb{I} - \hat{w} \otimes \hat{w}),$$

with $\alpha < 1$. In case of $\alpha < 0$ a unique smooth solution can be constructed (see [3], [4]). It would be interesting to consider a N-particle diffusion process with generator given by

(4.5.17), in which a is replaced by a_α. Of course now we expect a much better control on the limit $N \to \infty$ and, in particular, the propagation of chaos.

The Landau equation can also be obtained as a grazing collision limit from the homogeneous Boltzmann equation, for a sufficiently small α (see [1], [5], [3] and [4]). The case $\alpha = 1$ has been considered in [11].

In this paper we focus our attention on the Coulomb divergence $\alpha = 1$, which we think is the most physically relevant case. Indeed the Landau equation for $\alpha = 1$ is believed to hold in the so called weak-coupling limit, for Hamiltonian particle systems interacting by means of a smooth, short-range potential. See [2] and [10] for a formal derivation. Unfortunately up to now no rigorous result is known, even for short times.

4.6 Proof of Theorem 5

4.6.1 Preliminaries

In this section, we collect some preliminary properties satisfied by the N marginal distributions $f_j^N, j = 1, \dots, N$. In all this section N is fixed. We start by introducing some **Notations**. In the following, we will write

$$\mathrm{V}_j = (v_1, \dots, v_j), \quad \mathrm{V}_j^N = (v_{j+1}, \dots, v_N), \quad j = 1, \dots, N,$$

so that

$$f_j^N = f_j^N(\mathrm{V}_j, t) = \int dV_j^N \, W^N(\mathrm{V}_j, V_j^N, t).$$

Moreover,

$$a_{i,j} = a_{i,j}(V_N) = a(v_i - v_j), \quad P_{i,j} = P(v_i - v_j), \quad i, j = 1, \dots, N.$$

"$\cdot$" will denote the usual scalar product on $\mathbb{R}^3$, $\mathbb{R}^{3j}$ or $\mathbb{R}^{3N}$. For $V_N, \xi \in \mathbb{R}^{3N}$,

$$B(V_N) \cdot \xi = \begin{pmatrix} B_1(V_N) \cdot \xi \\ \cdot \\ \cdot \\ \cdot \\ B_N(V_N) \cdot \xi \end{pmatrix}$$

where $B_k(V_N) \in \mathbb{R}^{3N}$ is the k-th line of $B(V_N)$.

On the other hand, for $1 \leq k \leq N$ we will denote by

$$\nabla_{v_k} \cdot \xi = \sum_{i=1}^{3} \partial_{v_k^i} \xi_i,$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and $v_k = (v_k^1, v_k^2, v_k^3)$.

Finally, for every fixed j such that $1 \leq j \leq N$, for $1 \leq k, m \leq j$ we denote by

$$V_j^{k,m} = (v_1, \dots, v_{k-1}, v_m, v_{k+1}, \dots, v_{m-1}, v_k, v_{m+1}, \dots, v_j)$$

the vector obtained by exchanging the components v_k and v_m.

We start with an elementary property on the matrix B.

Lemma 7. *B is positive semi-definite, i. e. for all ξ*

$$(B \cdot \xi) \cdot \xi \geq 0.$$

More precisely, we have

$$(B \cdot \xi) \cdot \xi = \frac{1}{N} \sum_{i,j=1}^{N} \frac{|P_{i,j} \cdot (\xi_i - \xi_j)|^2}{|v_i - v_j|}, \quad \textit{where} \quad \xi = (\xi_i)_{1 \leq i \leq N}.$$

Proof. Fix $\xi \in \mathbb{R}^{3N}$, setting conventionally $a_{i,i} = 0$ for all i we get

$$\begin{aligned}(B \cdot \xi) \cdot \xi &= \sum_{i=1}^{N} \left(-\frac{1}{N} \sum_{j \neq i} a_{i,j} \cdot \xi_j + \frac{1}{N} \sum_{j} a_{i,j} \cdot \xi_i \right) \cdot \xi_i \\ &= \frac{1}{N} \sum_{i,j=1}^{N} \frac{P_{i,j} \cdot (\xi_i - \xi_j)}{|v_i - v_j|} \cdot \xi_i.\end{aligned}$$

Exchanging i and j in the sum we get, using that $P_{i,j}$ is a projector :

$$\begin{aligned}(B \cdot \xi) \cdot \xi &= \frac{1}{N} \sum_{i,j=1}^{N} \frac{P_{i,j} \cdot (\xi_i - \xi_j)}{|v_i - v_j|} \cdot (\xi_i - \xi_j) \\ &= \frac{1}{N} \sum_{i,j=1}^{N} \frac{|P_{i,j} \cdot (\xi_i - \xi_j)|^2}{|v_i - v_j|} \geq 0.\end{aligned}$$

□

Lemma 8. *Let $W^N(t)$ be the solution to eq.n (4.5.11). Then for any convex function $\Phi \in \mathcal{C}^2(\mathbb{R}^+; \mathbb{R})$, $\int dV_N \, \Phi(W^N)$ is decreasing in time; more precisely, we have*

$$\frac{d}{dt} \int dV_N \, \Phi(W^N(t)) = -\int dV_N \, \Phi''(W^N(t)) \nabla_{V_N} W^N \cdot (B^N \cdot \nabla_{V^N} W^N) \leq 0. \tag{4.6.1}$$

Proof. Look at

$$\partial_t W^N = L^N W^N.$$

Let us consider a convex function Φ, then

$$\begin{aligned}\frac{d}{dt} \int \Phi(W^N) &= \int dV_N \, \Phi'(W^N) \, \mathrm{div}_{V_N} (B^N \cdot \nabla_{V_N} W^N) \\ &= -\int dV_N \, \Phi''(W^N) \nabla_{V_N} W^N \cdot (B^N \cdot \nabla_{V_N} W^N).\end{aligned} \tag{4.6.2}$$

Taking into account the convexity of Φ and using Lemma 7 the r.h.s. of (4.6.2) is non positive and the statement of the Lemma holds. □

In particular, we will use Lemma 8 with $\Phi(x) = x\log(x)$. We denote by

$$S(W^N(t)) = \frac{1}{N}\int dV_N\, W^N(t)\log(W^N(t)) \tag{4.6.3}$$

the entropy per particle. In view of Lemma 8, $S(W^N(t))$ is decreasing in time

$$\frac{d}{dt}S(W^N(t)) = -\frac{1}{N}\int dV_N\, \frac{1}{W^N}\nabla_{V_N}W^N\cdot(B^N\cdot\nabla_{V_N}W^N) \le 0 \tag{4.6.4}$$

since $\Phi''(x) = 1/x \ge 0$. In what follows we will use the explicit formula for the entropy production:

$$\begin{aligned} -\frac{d}{dt}S(W^N(t)) = &\frac{1}{N^2}\sum_{k,l=1}^{N}\int dV_N\,\frac{|P_{k,l}\cdot\left[\nabla_{v_k}W^N-\nabla_{v_l}W^N\right]|^2}{W^N|v_k-v_l|_N}\\ &+\frac{1}{N^2}\int dV_N\,\frac{1}{W^N}|\nabla_{V_N}W^N|^2. \end{aligned} \tag{4.6.5}$$

Remark 4. *Although the entropy $S(W^N(t))$ decreases,*

$$S(f_j^N(t)) \equiv \frac{1}{j}\int f_j(t)\log(f_j(t))$$

is not decreasing in general. However by subadditivity of the entropy we know (see e.g. [7]) that

$$S(f_j^N(t)) \le S(W^N(t)) \tag{4.6.6}$$

so that

$$S(f_j^N(t)) \le C \tag{4.6.7}$$

since we have $S(W^N(0)) \le C$.

Remark 5. *In case of factorization, i. e. $f_j = f^{\otimes j}$, we have the equality*

$$S(f_j) = S(f). \tag{4.6.8}$$

Eq.n (4.6.5) provides a useful estimate given by the following

Corollary 1. *Let $0 \le s_1 \le s_2$. Then*

$$\sum_{k,l=1}^{N}\int_{s_1}^{s_2}ds\int dV_N\,\frac{|P_{k,l}\cdot\left[\nabla_{v_k}W^N-\nabla_{v_l}W^N\right]|^2}{W^N|v_k-v_l|_N} \le CN^2.$$

Remark 6. *Due to the symmetry of W^N, all terms of the above sum are equal and hence each term is bounded uniformly in N, namely for all $1 \le k, l \le N$*

$$\int ds\int dV_N\,\frac{|P_{k,l}\cdot\left[\nabla_{v_k}W^N-\nabla_{v_l}W^N\right]|^2}{W^N|v_k-v_l|_N} \le C. \tag{4.6.9}$$

4.6.2 Basic estimates

Proposition 1. *Let $1 \le j \le N-1$ and $\varphi \in C_c^2(\mathbb{R}^{3j}, \mathbb{R})$ be a test function. Let $0 \le s_1 \le s_2$. Then*

$$\int_{s_1}^{s_2} ds \left| \int dV_j \, L_j^N \, f_j^N(V_j)\varphi(V_j) \right| \le \frac{C(\varphi) j^2}{N} |s_1 - s_2|^{1/2}$$

and

$$\int_{s_1}^{s_2} ds \left| \int dV_j \, C_{j+1}^N \, f_{j+1}^N(V_j)\varphi(V_j) \right| \le C(\varphi) j |s_1 - s_2|^{1/2},$$

where $C(\varphi)$ depends only on φ and on the initial data, but not on N.

Proof. We begin by estimating C_{j+1}^N. Recall (4.5.13). By integrating by parts, we have

$$\begin{aligned}
&\int dV_j \, C_{j+1}^N \, f_{j+1}^N(V_j)\varphi(V_j) \\
&\qquad = -\sum_{k=1}^{j} \int dV_j \, dV_j^N \, a^N(v_k - v_{j+1}) \cdot (\nabla_{v_k} W^N - \nabla_{v_{j+1}} W^N)(V_j, \mathrm{V}_j^N) \cdot \nabla_{v_k}\varphi(V_j) \\
&\qquad = \frac{1}{2} \sum_{k=1}^{j} \int dV_N \, a^N(v_k - v_{j+1}) \cdot (\nabla_{v_k} W^N - \nabla_{v_{j+1}} W^N)(V_N) \cdot \\
&\qquad (\nabla_{v_k}\varphi(V_j) - \nabla_{v_k}\varphi(V_j^{k,j+1})),
\end{aligned}$$

where

$$V_j^{k,j+i} = (v_1, \dots, v_{k-1}, v_{j+1}, v_{k+1}, \dots, v_j)$$

and we exchanged variables v_k and v_{j+1} in the second line and used the symmetry of W^N.

Therefore

$$\begin{aligned}
&\int_{s_1}^{s_2} ds \Big| \int dV_j \, C_{j+1}^N \, f_{j+1}^N(V_j)\varphi(V_j) \Big| \\
&\qquad \le \frac{1}{2} \int_{s_1}^{s_2} ds \sum_{k=1}^{j} \int dV_N \, \frac{\sqrt{W^N}}{\sqrt{W^N}} \frac{|\nabla_{v_k}\varphi(V_j) - \nabla_{v_k}\varphi(V_j^{k,j+1})|}{\sqrt{|v_k - v_{j+1}|_N}} \\
&\qquad \frac{|P_{k,j+1} \cdot (\nabla_{v_k} W^N - \nabla_{v_{j+1}} W^N)(V_N)|}{\sqrt{|v_k - v_{j+1}|_N}};
\end{aligned}$$

(by using the Cauchy-Schwarz inequality)

$$\begin{aligned}
&\le \frac{1}{2} \sum_{k=1}^{j} \left(\int_{s_1}^{s_2} ds \int dV_N \, W^N(V_N) \frac{|\nabla_{v_k}\varphi(V_j) - \nabla_{v_k}\varphi(V_j^{k,j+1})|^2}{|v_k - v_{j+1}|_N} \right)^{1/2} \cdot \\
&\left(\int_{s_1}^{s_2} ds \int dV_N \, \frac{|P_{k,j+1}(\nabla_{v_k} W^N(V_N) - \nabla_{v_{j+1}} W^N(V_N))|^2}{W^N(V_N)|v_k - v_{j+1}|_N} \right)^{1/2}.
\end{aligned}$$

By virtue of mean-value Theorem applied to $\nabla_{v_k}\varphi$ and (4.6.9) we get the bound on C_{j+1}^N:

$$\int_{s_1}^{s_2} ds \int C_{j+1}^N f_{j+1}^N \varphi(V_j) \, dV_j \le j \, C(\varphi) |s_1 - s_2|^{1/2}. \tag{4.6.10}$$

By performing exactly the same computations we are led to

$$\begin{aligned}
&\int_{s_1}^{s_2} ds \left| \int dV_j \, L_j^N f_j^N(V_j)\varphi(V_j) \right| \\
&\qquad \le \frac{C(\varphi)}{N} \sum_{\substack{k\neq l \\ k,l=1}}^{j} \left(\int_{s_1}^{s_2} ds \int dV_N \frac{|P_{k,l}\cdot\left[\nabla_{v_k} W^N(V_N) - \nabla_{v_l} W^N(V_N)\right]|^2}{W^N|v_k - v_l|_N} \right)^{1/2} \\
&\qquad \le \frac{C(\varphi) j^2}{N} |s_1 - s_2|^{1/2}.
\end{aligned}$$

The proof is now complete. □

4.6.3 Convergence

In this subsection, we establish the weak compactness for the f_j^N by making use of the uniform estimates established in the previous subsection.

Proposition 2. *Let f_j^N satisfy the hierarchy* (4.5.12). *There exists a subsequence $N_k \to +\infty$ such that for any fixed j, there exists $f_j = f_j(V_j, t) \in C([0,T]; \mathcal{M}(j))$, with finite energy and entropy, such that $f_j^{N_k}$ converges to f_j weakly in the sense of measures, locally uniformly in time.*

Proof. We fix j. For $\varphi \in C_c(\mathbb{R}^{3j})$, we set

$$t \mapsto g_\varphi^N(t) = \int dV_j \, f_j^N(V_j, t)\varphi(V_j).$$

We obtain a uniformly bounded sequence of functions on $\mathbb{R}_+$. Moreover, when $\varphi \in C_c^2(\mathbb{R}^{3j})$, by virtue of the proof of Proposition 1 the function g_φ^N is uniformly equicontinuous. Hence, by Ascoli's theorem and density of $C_c^2(\mathbb{R}^{3j})$ in $C_c(\mathbb{R}^{3j})$, there exists a subsequence N_k such that for all $\varphi \in C_c(\mathbb{R}^{3j})$, $g_\varphi^{N_k}$ converges locally uniformly in time to some function $g_\varphi(t)$. Now, for each fixed t, the map

$$\varphi \mapsto g_\varphi(t)$$

is a positive linear form on $C_c(\mathbb{R}^{3j})$. Thus the Riesz representation theorem ensures the existence of a measure $df_j(t)$ such that $g_\varphi(t) = \int \varphi df_j(t)$. On the other hand, $(f_j^N)(t)$ has uniformly bounded entropy and energy; therefore it is weakly relatively compact in L^1. This shows that in fact $df_j(t) = f_j(t)\, dV_j$ is an absolutely continuous probability measure and has finite entropy and energy. This concludes the proof of the proposition.

□

4.6.4 End of the proof

We are now in position to complete the proof of Theorem 5. We fix $j \geq 0$. For any $g \in C_c^2(\mathbb{R}^{3(j+1)})$ we set

$$
\begin{aligned}
C_{j+1}^{\delta} g(V_j) &= \sum_{k=1}^{j} \nabla_{v_k} \cdot \int [(1-\chi_\delta) a](v_k - v_{j+1}) \cdot (\nabla_{v_k} g - \nabla_{v_{j+1}} g)(V_j, v_{j+1})\, dv_{j+1}, \\
\overline{C}_{j+1}^{\delta} g(V_j) &= \sum_{k=1}^{j} \nabla_{v_k} \cdot \int [\chi_\delta a](v_k - v_{j+1}) \cdot (\nabla_{v_k} g - \nabla_{v_{j+1}} g)(V_j, v_{j+1})\, dv_{j+1},
\end{aligned}
$$

so that

$$
C_{j+1}(g) = C_{j+1}^{\delta}(g) + \overline{C}_{j+1}^{\delta}(g). \tag{4.6.11}
$$

The analogous decomposition holds for C_{j+1}^N:

$$
C_{j+1}^N = C_{j+1}^{N,\delta} + \bar{C}_{j+1}^{N,\delta}
$$

where a^N replaces a in (4.6.11). Note that $C_{j+1}^{N,\delta} = C_{j+1}^{\delta}$ whenever N is sufficiently large.

We will show that for all $t \geq 0$ and for all test function φ in C_c^2 we have

$$
\begin{aligned}
&\int_0^t ds \int dV_j\, C_{j+1}^{N,\delta} f_{j+1}^N \varphi \\
&\qquad = \int_0^t ds \int dV_j\, C_{j+1}^{\delta} f_{j+1}^N \varphi \longrightarrow \int_0^t ds \int dV_j\, C_{j+1}^{\delta} f_{j+1} \varphi
\end{aligned} \tag{4.6.12}
$$

when $N \to \infty$ and

$$
\sup_{N \geq j} \left| \int_0^t ds \int dV_j\, \overline{C}_{j+1}^{N,\delta} f_{j+1}^N \varphi \right| \leq C(\varphi) \delta^{1/2}. \tag{4.6.13}
$$

First, (4.6.12) follows by the convergence established in Proposition 2 and by two integrations by parts.

As regards (4.6.13), we need a symmetrized form as in the proof of Proposition 1. Mimicking the computations of Proposition 1 we find

$$
\begin{aligned}
&\left| \int_0^t ds \int dV_j\, \overline{C}_{j+1}^{N,\delta} f_{j+1}^N \varphi \right| = \left| \int_0^t ds \int dV_j\, \overline{C}_{j+1}^{\delta} f_{j+1}^N \varphi \right| \\
&\leq C \sum_{k=1}^{j} \left(\int_0^t ds \int dV_N \, \frac{|P_{k,j+1} \cdot (\nabla_{v_k} W^N - \nabla_{v_{j+1}} W^N)|^2}{W^N |v_k - v_{j+1}|_N} \right)^{1/2} \\
&\quad \left(\int_0^t ds \int dV_N\, \chi_\delta^2(|v_k - v_{j+1}|) \frac{|\nabla_{v_k} \varphi(V_j) - \nabla_{v_k} \varphi(V_j^{k,j+1})|^2}{|v_k - v_{j+1}|} W^N \right)^{1/2}.
\end{aligned}
$$

Applying once more inequality (4.6.9), the first term in the right-hand side is bounded. Next, we observe that in view of the support properties of χ_δ, the mean-value theorem yields

$$
\chi_\delta^2(|v_k - v_{j+1}|) |\nabla_{v_k} \varphi(V_j) - \nabla_{v_k} \varphi(V_j^{k,j+1})|^2 \leq C \delta |v_k - v_{j+1}|.
$$

Finally we obtain

$$\left| \int_0^t ds \int dV_j \, \overline{C}_{j+1}^{\delta} f_{j+1}^N \varphi \right| \leq C\delta^{1/2},$$

and (4.6.13) follows. Hence the proof of Theorem 5 is complete.

We conclude this section with some comments concerning additional regularity for the marginal f_j^N. In fact, the control on the production of the total entropy (see Corollary 1) yields a uniform control on the gradients of f_j^N. More precisely, we have for all $1 \leq k, l \leq j$

$$\int ds \int dV_j \, \frac{|P_{k,l} \cdot \left(\nabla_{v_k} f_j^N - \nabla_{v_l} f_j^N\right)|^2}{f_j^N |v_k - v_l|_N} \leq C. \tag{4.6.14}$$

Indeed, we have

$$\begin{aligned}
&\int ds \int dV_j \, \frac{|P_{k,l} \cdot \left(\nabla_{v_k} f_j^N - \nabla_{v_l} f_j^N\right)|^2}{f_j^N |v_k - v_l|_N} \\
&= \int ds \int dV_j \, \frac{1}{f_j^N |v_k - v_l|_N} \left| \int P_{k,l} \cdot \left(\nabla_{v_k} W^N - \nabla_{v_l} W^N\right) \, dV_j^N \right|^2 \\
&= \int ds \int dV_j \, \frac{f_j^N}{|v_k - v_l|_N} \left| \int P_{k,l} \cdot \left(\nabla_{v_k} W^N - \nabla_{v_l} W^N\right) \frac{1}{W^N} \, \frac{W^N}{f_j^N} \, dV_j^N \right|^2 \\
&\leq \int ds \int dV_N \, \frac{f_j^N}{|v_k - v_l|_N} \int |P_{k,l} \cdot \left(\nabla_{v_k} W^N - \nabla_{v_l} W^N\right)|^2 \frac{1}{(W^N)^2} \, \frac{W^N}{f_j^N} \\
&= \int ds \int dV_N \, \frac{|P_{k,l} \cdot \left(\nabla_{v_k} W^N - \nabla_{v_l} W^N\right)|^2}{W^N |v_k - v_l|_N},
\end{aligned}$$

where we have applied Jensen's inequality in the last inequality. The conclusion follows from (4.6.9).

In particular, (4.6.14) implies that

$$\frac{P_{k,l}}{|v_k - v_l|_N} \cdot \left(\nabla_{v_k} \sqrt{f_j^N} - \nabla_{v_l} \sqrt{f_j^N}\right)$$

is bounded in $L^2(\mathbb{R}_+ \times \mathbb{R}^{3j})$; hence, following the same arguments as in [11] we can conclude that

$$\frac{P_{k,l}}{|v_k - v_l|} \cdot \left(\nabla_{v_k} \sqrt{f_j} - \nabla_{v_l} \sqrt{f_j}\right) \in L^2(\mathbb{R}_+ \times \mathbb{R}^{3j}), \tag{4.6.15}$$

so that one can use the symmetrized form already used in the proof of Proposition 1 to define $C_{j+1} f_{j+1}$ as in [11]:

$$\begin{aligned}
&\int ds \int dV_j \, C_{j+1} f_{j+1} \varphi \\
&= -\frac{1}{2} \sum_{k=1}^{j} \int ds \int dV_j \, a_{k,j+1} \cdot \left(\nabla_{v_k} f_{j+1} - \nabla_{v_{j+1}} f_{j+1}\right) \cdot \left(\nabla_{v_k} \varphi(V_j) - \nabla_{v_k} \varphi(V_j^{k,j+1})\right).
\end{aligned}$$

Acknowledgements

One of the authors (MP) thanks L. Desvillettes for useful and illuminating discussions.

Bibliography

[1] A. A. Arseniev and O. E. Buryak, *On a connection between the solution of the Boltzmann equation and the solution of the Landau-Fokker-Planck equation,* (Russian), Mat. Sb., **181** (1992), 435–446; translation in Math. USSR-Sb. **69** (1991), 465–478.

[2] R. Balescu, "Equilibrium and Nonequilibrium Statistical Mechanics," John Wiley & Sons, New-York, 1975.

[3] L. Desvillettes and C. Villani, *On the spatially homogeneous Landau equation for hard potentials. II. H-theorem and applications,* Comm. Partial Differential Equations **25** (2000), 261–298.

[4] L. Desvillettes and C. Villani, *On the spatially homogeneous Landau equation for hard potentials. I. Existence, uniqueness and smoothness,* Comm. Partial Differential Equations **25** (2000), 179–259.

[5] T. Goudon, *On Boltzmann equations and Fokker-Planck asymptotics: influence of grazing collisions,* J. Statist. Phys. **89** (1997), 751–776.

[6] M. Kac, *Foundations of kinetic theory*, in "Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability", University of California Press, Berkeley and Los Angeles (1956).

[7] A. I. Khinchin, "Mathematical Foundations of Information Theory," New York: Dover 1957.

[8] E. M. Lifshitz and L. P. Pitaevskii, "Course of theoretical physics Landau-Lifshits," Pergamon Press, Oxford-Elmsford, N.Y. 1981.

[9] R. Peyre, *Some ideas about quantitative convergence of collision models to their mean field limit*, J. Stat. Phys. **136** (2009), 1105–1130.

[10] M. Pulvirenti, *The weak-coupling limit of large classical and quantum systems*, (English summary) in "International Congress of Mathematicians", Eur. Math. Soc., Zürich (2006).

[11] C. Villani, *On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations*, Arch. Rational Mech. Anal., **143** (1998), 273–307.

Chapter 5

Mean–Field limit: the Vlasov–Poisson system

Sections 5.3–5.4 of the present Chapter are extracted from [DMS].

5.1 Introduction

Both from mathematical and physical point of view, one of most important feature to analyze is the evolution of the density function f in presence of Coulomb interaction between charged particles. The basic model describing this phenomenum is given by the Vlasov–Poisson equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 \\ E(t,x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(t,y)\, dy\,, \\ \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\, dv\ , \end{cases} \tag{5.1.1}$$

where $f(t,x,v) \geq 0$ is defined on the phase space $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and it denotes a density of electric particles, called a plasma, subjected to a self-induced electric force field $E(t,x)$.
It is a good model to describe a plasma on a short time scale[1] (see [Bal] or [LL]); on the other hand, if we want to study it in a long time interval, we have to take into account collisions among particles, so that it is natural to replace the r.h.s. of (5.1.1) by the Boltzmann collision operator. As already observed in Chapter 4, the problem is that the Boltzmann collision operation does not make sense when the interaction between particles is of Coulomb type; indeed -even for very regular density functions- the collision integral is infinite. This justifies the passage to the Landau eq.n, as pointed out in Section 4.3.
Among the many problems linked to the Vlasov–Poisson equation, we list the existence, uniqueness and regularity of the solution; the derivation of (5.1.1) from particle systems[2]; the

[1] Indeed, it does not take into account the collisions among particles.

[2] See Section 5.2 for an idea of the proof when the interaction potential is smooth, namely in the case of

understanding of the celebrated Landau damping ([L-36], [CM] and [MV]). In what follows we are interested in the well–posedness problem for the Vlasov–Poisson equation and for a slightly different model: the plasma–charge model (see eq.ns (5.3.1) in Section 5.3). We mention that the Cauchy problem for the Vlasov-Poisson system (5.3.1) has been object of a large variety of works in the last decades (we refer to Section 5.3.1 for references in this respect).

The Chapter is organized as follows. In the next Section we give an idea of derivation problem for a smooth, compactly supported potential, pointed out the relevance of the mean–field limit. Section 5.3 is devoted to the proof of Theorem 6. The general procedure, which follows the lines of [LP], consists in deriving a priori estimates for the moments of a sequence of smooth solutions to (5.3.1)–(5.3.3) obtained by regularizing the initial density in order to obtain a global solution by compactness arguments.
In Subsection 5.3.3 we gather some basic facts and a priori estimates for the modified Vlasov-Poisson system (5.3.1). We also derive some first estimates for the energy moments. In Subsection 5.3.4 we introduce a notion of almost-free flow, which enables to express the solution of (5.3.1) by means of Duhamel's formula with a suitable source term. Then, in Subsection 5.3.5 we establish intermediate a priori estimates for the moments, which as a byproduct ensure that the moments are uniformly bounded for small times. These estimates are exploited to show that the moments are uniformly bounded for all times in Subsection 5.3.6. They provide a global solution satisfying the assumptions of Theorem 6 by standard compactness arguments. An appendix is also devoted to the proof of technical estimates on the almost-free flow.

5.2 From particle system to the Vlasov equation: heuristic derivation

We consider N interacting particles in the whole space as in Section 1.2, whose dynamics is described by the Newton equations (1.2.3). Since we are interested in a *mean–field* description of the system, we rescale the system in order to obtain a weak and long range interaction on the same scale of time; this means that we perform the following scaling:

$$\Phi(x_i - x_j) \longrightarrow \frac{1}{N}\Phi(x_i - x_j)\ . \tag{5.2.1}$$

---

the Vlasov equation, i.e.

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 \\ E(t,x) = \int_{\mathbb{R}^3} \nabla\Phi(x-y)\rho(t,y)\,dy\,, \\ \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv\ , \end{cases} \tag{5.1.2}$$

where $\Phi \in \mathcal{C}_b^2(\mathbb{R}^3)$. In the case of the Vlasov–Poisson system, the problem is completely open.

The variables are $(\mathbf{x}_N, \mathbf{v}_N) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$ and they describe positions and velocities of the N particles interacting by means of a weak potential of order $O\left(\frac{1}{N}\right)$, following the Newton equations (1.2.3). We fix the initial configuration $(\mathbf{x}_N, \mathbf{v}_N) \in (\mathbb{R}^{3N} \times \mathbb{R}^{3N})$, where $(x_i, v_i) \in (\mathbb{R}^3 \times \mathbb{R}^3)$ is a point in the one–particle phase space; the corresponding Hamiltonian is given by

$$H(\mathbf{x}_N, \mathbf{v}_N) = \sum_{i=1}^{N} \frac{|v_i|^2}{2} + \frac{1}{N} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \Phi(x_i - x_j) \; . \tag{5.2.2}$$

The factor $\frac{1}{N}$ in front of the interaction potential is the expression of the mean–field character of the Hamiltonian, since it guaranties that, in the limit of N large, both kinetic and potential energy are of the same magnitude order. Indeed the total kinetic energy is given by a sum of N terms and the total potential energy is a sum of $\frac{N(N-1)}{2}$ terms, so that the factor $\frac{1}{N}$ in front of the potential energy in (5.2.2) is such that, when the number of particles becomes huge, the kinetic and the potential energy are of order $\mathcal{O}(N)$. We observe that, in the limit $N \to +\infty$, thanks to the mean–field scaling, all particles are interacting with each other so that the interaction is long range, but weak.
The aim of this Section is to derive, at least heuristically in the limit of N large, the Vlasov equation, starting from the above N–particle system. To this end, we introduce $\mathcal{M}(\mathbb{R}^3 \times \mathbb{R}^3)$, the space of measures on the one–particle phase space. To simplify the notation we denote by $\mathbf{z}_N$ the points in the N–particle phase space, i.e. $\mathbf{z}_N = (\mathbf{x}_N, \mathbf{v}_N) \in (\mathbb{R}^{3N} \times \mathbb{R}^{3N})$ with $z_i = (x_i, v_i) \in \mathbb{R}^3$. On $\mathcal{M}(\mathbb{R}^3 \times \mathbb{R}^3)$ we define the empirical measure associated to the N–particle configuration $\mathbf{z}_N$ by

$$\mu^N(z; \mathbf{z}_N) = \frac{1}{N} \sum_{i=1}^{N} \delta(z - z_i), \quad \forall z \in (\mathbb{R}^3 \times \mathbb{R}^3) \tag{5.2.3}$$

where $\delta(\cdot)$ is the Dirac measure on $(\mathbb{R}^3 \times \mathbb{R}^3)$. We observe that μ^N is a probability measure on $(\mathbb{R}^3 \times \mathbb{R}^3)$, i.e. $\mu^N \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)$, which is an infinite dimensional space on the one–particle phase space, depending on the configuration $\mathbf{z}_N$ in the N–particle phase space. More precisely, to each configuration $\mathbf{z}_N$, we can associate the empirical measure $\mu^N(z; \mathbf{z}_N)$ which counts the number of particles in the phase space. In the sequel, we will write $\mu^N(z)$ instead of $\mu^N(z; \mathbf{z}_N)$, omitting the dependence on $\mathbf{z}_N$ when not misleading. We notice that knowing the time evolution $\mu_t^N(z; \mathbf{z}_N)$ is equivalent to determine the trajectory of each particle. Since we are interested in a statistical description, we assume that there exists a probability distribution f_0^N on the N–particle phase space such that the initial configuration $\mathbf{z}_N$ is distributed according to the factorized measure $f_0^N(\mathbf{z}_N) = f_0^N(\mathbf{x}_N, \mathbf{v}_N)$; in other words the probability density can be written as follows:

$$f_0^N(\mathbf{x}_N, \mathbf{v}_N) = \prod_{i=1}^{N} f_0(x_i, v_i) = f_0^{\otimes N} \; , \tag{5.2.4}$$

where f_0 is a regular function on the one–particle phase space. Let μ_0^N be the empirical measure associated to $\mathbf{z}_N$, we define $\mu_t^N = \mu_t^N(z; \mathbf{z}_N)$ the empirical measure associated to the time evolution of the configuration $\mathbf{z}_N$ described by the rescaled Newton equations

$$\begin{cases} \dot{x}_i(t) = v_i(t) \ , \\ \\ \dot{v}_i(t) = -\frac{1}{N} \sum_{\substack{j=1 \\ j\neq i}}^N \nabla_{x_i} \Phi(x_i(t) - x_j(t)) \ , \quad i = 1, \dots, N \end{cases} \tag{5.2.5}$$

with initial datum $\mathbf{z}_N$.
We denote by $\mathcal{C}_c^\infty(\mathbb{R}^3)$ the space of test function with compact support. We notice that if the interaction potential is regular, namely $\Phi \in \mathcal{C}_b^2(\mathbb{R}^3)$, for all $\varphi \in \mathcal{C}_c^\infty$ the following relation holds:

$$\begin{aligned} (\varphi, \mu_t^N) = & \int_{\mathbb{R}^3} dz \mu_t^N(z; \mathbf{z}_N(t)) \varphi(z) = \\ & = \frac{1}{N} \sum_{i=1}^N \varphi(z_i(t)) \ . \end{aligned} \tag{5.2.6}$$

Thanks to the regularity hypothesis on the potential, we can compute the time derivative of (5.2.6) using the Newton equations (5.2.5)

$$\begin{aligned} \frac{d}{dt}(\varphi, \mu_t^N) = & \frac{1}{N} \sum_{i=1}^N \frac{d}{dt} \varphi(z_i(t)) = \\ & = \frac{1}{N} \sum_{i=1}^N \dot{x}_i \cdot \nabla_{x_i} \varphi(x_i(t), v_i(t)) + \frac{1}{N} \sum_{i=1}^N \dot{v}_i \cdot \nabla_{v_i} \varphi(x_i(t), v_i(t)) = \\ & = \frac{1}{N} \sum_{i=1}^N v_i \cdot \nabla_{x_i} \varphi(x_i(t), v_i(t)) - \frac{1}{N^2} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \nabla_{x_i} \Phi(x_i(t) - x_j(t)) \cdot \nabla_{v_i} \varphi(x_i(t), v_i(t)) = \\ & = (v \cdot \nabla_x \varphi, \mu_t^N) - ((\nabla_x \Phi * \mu_t^N) \cdot \nabla_v \varphi, \mu_t^N) \ . \end{aligned} \tag{5.2.7}$$

Now we write explicitly the second term in the r.h.s. of (5.2.7):

$$\begin{aligned} (\nabla \Phi * \mu_t^N)(x) = & \int dy \int dw \nabla_x \Phi(x - y) \left(\frac{1}{N} \sum_{j=1}^N \delta(y - x_j(t)) \delta(w - v_j(t)) \right) = \\ & = \frac{1}{N} \sum_{j=1}^N \int dy \nabla_x \Phi(x - y) \delta(y - x_j(t)) = \\ & = \frac{1}{N} \sum_{j=1}^N \nabla_x \Phi(x - x_j(t)) \ . \end{aligned} \tag{5.2.8}$$

This means that μ_t^N is a solution (in the sense of measures) to the weak formulation of the Vlasov equation

$$\partial_t \mu_t^N + v \cdot \nabla_x \mu_t^N = (\nabla_x \Phi * \mu_t^N) \cdot \mu_t^N \tag{5.2.9}$$

with initial datum μ_0^N.
The Strong Law of Large Numbers ensures us that, if the initial datum verifies condition (5.2.4), then

$$\mu_0^N \longrightarrow f_0 \tag{5.2.10}$$

in the limit $N \to \infty$. The above limit holds in the weak–topology.
If we assume that μ_t^N is a weak solution to the Vlasov equation and that the interaction potential is regular enough (i.e. $\Phi \in \mathcal{C}_b^2(\mathbb{R}^3)$), then we can use the Dobrushin stability result (see [D-79]), stating that if μ_t^1 and μ_t^2 are two solutions of the Vlasov equation with initial data μ_0^1 and μ_0^2 respectively, and if $\Phi \in \mathcal{C}_b^2(\mathbb{R}^3)$, there exists a constant C, depending only on the potential, such that

$$\mathcal{W}(\mu_t^1, \mu_t^2) \leq e^{Ct}\mathcal{W}(\mu_0^1, \mu_0^2) \;, \tag{5.2.11}$$

where $\mathcal{W}$ is the standard Wasserstein distance.
Using (5.2.11), we can prove that, in the weak topology,

$$\mu_t^N \longrightarrow f(t) \tag{5.2.12}$$

in the limit $N \to \infty$, where $f(t)$ is the strong solution to the Vlasov equation with initial condition f_0.
If the potential is not $\mathcal{C}_b^2(\mathbb{R}^3)$ very little is known about the rigorous derivation of the Vlasov equation. In particular the case of Coulomb potential, i.e. the derivation of the Valsov–Poisson system from particles, is completely open. However, some progresses have been done in this direction by Hurray and Jabin [HJ], who solved the problem when the gradient of the interaction potential is given by $\nabla\Phi(x) \sim \frac{1}{|x|^\alpha}$, with α strictly less then one.

In the following Sections we report a preprint written in collaboration with L. Desvillettes and E. Miot, [DMS].

5.3 An existence result for the 3d repulsive plasma–charge model

5.3.1 Introduction

The purpose of this paper is to study the following three dimensional Vlasov-Poisson system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E + F) \cdot \nabla_v f = 0 \\ E(t,x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(t,y)\, dy, \\ \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\, dv, \\ F(t,x) = \frac{x-\xi(t)}{|x-\xi(t)|^3}. \end{cases} \tag{5.3.1}$$

Here $f(t,x,v) \geq 0$ is defined on the phase space $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and it denotes a density of electric particles, called a plasma, subjected to a self-induced electric force field $E(t,x)$. The plasma interacts with a point charge, located at $\xi(t)$ with velocity $\eta(t)$, which induces the singular electric field $F(t,x)$. The evolution of the charge is itself given by

$$\begin{cases} \dot{\xi}(t) = \eta(t), \\ \dot{\eta}(t) = E(t, \xi(t)). \end{cases} \tag{5.3.2}$$

The initial conditions associated to (5.3.1)-(5.3.2) are

$$(\xi(0), \eta(0)) = (\xi_0, \eta_0), \quad f(0,x,v) = f_0(x,v). \tag{5.3.3}$$

The main result of this Section may be formulated as follows:

Theorem 6. *Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ be nonnegative, let $(\xi_0, \eta_0) \in \mathbb{R}^3 \times \mathbb{R}^3$, be such that*

(i) $\mathcal{M}_0 = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x,v)\, dx\, dv < 1$;

(ii) There exists $m_0 > 6$ such that for all $m < m_0$

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(|v|^2 + \frac{1}{|x - \xi_0|} \right)^{m/2} f_0(x,v)\, dx\, dv < +\infty.$$

Then there exists a global weak solution (f, ξ) to the system (5.3.1)–(5.3.3), *with $f \in C(\mathbb{R}_+, L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) - w^*)$ and $\xi \in C^2(\mathbb{R}_+)$.*
Moreover, for all $t \in \mathbb{R}_+$ and for all $m < \min(m_0, 7)$,

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(|v|^2 + \frac{1}{|x - \xi(t)|} \right)^{m/2} f(t,x,v)\, dx\, dv < +\infty.$$

Remark 7. In fact one is able to obtain a polynomial in time growth on the moments (see later):

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(|v|^2 + \frac{1}{|x - \xi(t)|} \right)^{m/2} f(t,x,v)\, dx\, dv \leq C(m,t) < +\infty,$$

where $C(m,t)$ is polynomial in t.

The Cauchy problem for the Vlasov-Poisson system (5.3.1), with or without point charge, has been the object of a large variety of works in the last decades. For the purely Vlasov-Poisson system without charge, namely $F = 0$, global existence and uniqueness of classical solutions where obtained by Ukai and Okabe [9] in two dimensions. The three dimensional case is more delicate and requires more care. Global weak solutions with finite energy were first built by Arsenev [2] but uniqueness is not known to hold in that class.

Then, global existence and, in some cases, uniqueness, of more regular solutions where established separately by Lions and Perthame [7] and by Pfaffelmoser [11] by means of different techniques. In both works the main issue consists in controlling the large plasma velocities for all time in order to propagate regularity properties of the solution.
In [7], this is achieved by constructing weak solutions with finite velocity moments of order higher than three,

$$\iint |v|^m f(t,x,v)\,dx\,dv < \infty, \quad m > 3,$$

which, by Sobolev embeddings, implies further bounds on the spatial density and on the electric field. In particular, if the solution admits finite moments of order $m > 6$ then the electric field is uniformly bounded and uniqueness holds under some additional regularity assumptions on the initial density. On the other hand, the theory of Di Perna and Lions [4] ensures that such solutions are transported by characteristics which are defined in a weak sense. In contrast with the eulerian approach of [7], the strategy of [11] relies on a careful analysis of the characteristics to control the growth of the velocity support and thereby obtain global existence and uniqueness of classical compactly supported solutions, which moreover propagate the regularity of the initial condition.
We refer to the further improvements and developments by Schaeffer [13], Wollman [14], Gasser, Jabin and Perthame [5] and Loeper [8]. Finally, Pallard [10] recently combined eulerian and lagrangian points of view to establish existence of solutions propagating velocity moments larger than two.

The study of the modified Vlasov-Poisson system with macroscopic point charges was initiated more recently by Caprino and Marchioro [3]. In two dimensions, they proved global existence and uniqueness of solutions à la Pfaffelmoser. This was then extended to the three-dimensional case by Marchioro, Miot and Pulvirenti [12]. The results of [3] and [12] hold for initial plasma densities that do not overlap the charge. Thanks to the repulsive nature of the plasma-charge interaction, this property remains true at later times so that the field induced by the charge is bounded on the support of the density and the velocities of the plasma particles do not blow up. The analysis of [3] and [12] exploit the notion of energy, defined in this context by

$$h(t,x,v) = \frac{|v-\eta(t)|^2}{2} + \frac{1}{|x-\xi(t)|}.$$

It turns out that the variation of the energy along the plasma characteristics is controlled by the electric field, exactly as that of the velocity in the absence of charge. On the other hand, the energy controls both the velocity and the distance to the charge. This makes it possible to adapt Pfaffelmoser's arguments by replacing the notion of largest velocity of the plasma particles by that of the largest energy $\sup_{\mathrm{supp}(f(t))} h$, which by assumption is initially finite. Unfortunately, when the plasma density overlaps the charge, the energy is not bounded and this method fails. In order to treat densities with unbounded energy, which is the purpose of the present paper, we adapt the PDE point of view from [7], and we show existence of a

solution propagating the energy moments (see Definition 1 hereafter). In particular, since the energy moments control the velocity moments, we recover all additional regularity properties on the electric field which have been established in [7]. We emphasize that Theorem 6 allows for initial densities that do not necessarily vanish in a neighborhood of the charge but that have to decay close to it in some sense; unfortunately it does not include the "generic" densities that are constant close to the charge.
On the other hand, we mention that our techniques do not enable to obtain uniqueness because of the singularity of F in the neighborhood of the charge. Finally, we believe that the limitation $m_0 < 7$ (appearing in the proof of Proposition 14) is purely technical. We also hope to extend Theorem 6 to the case of several point charges being all positively charged, as is the case in [3] and [12].

Another situation that could also be addressed is the one where the charge is kept fixed (for example at the origin). Then the analog of Theorem 6 can be obtained without the condition (i) and the condition $m_0 > 6$ can be replaced by $m_0 > 3$. In this latter case the electric field is not uniformly bounded, and we are not able to prove the existence of characteristics along which the density is constant. We do not provide the details here.

Thanks to the estimates proved in Theorem 6, it will turn out that, as in [7], one can define a notion of flow lines along which the density is constant. More precisely, there exists a map $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \{\xi_0\} \times \mathbb{R}^3 \mapsto (\boldsymbol{x}(t,x,v), \boldsymbol{v}(t,x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $f(t) = (\boldsymbol{x}(t), \boldsymbol{v}(t))_\# f_0$ for all $t \in \mathbb{R}_+$, and such that

(i) For all $(x,v) \in \mathbb{R}^3 \setminus \{\xi_0\} \times \mathbb{R}^3$, $t \mapsto (\boldsymbol{x}(t,x,v), \boldsymbol{v}(t,x,v) \in C^1(\mathbb{R}_+)$ is a solution of

$$\begin{cases} \dot{\boldsymbol{x}}(t,x,v) = \boldsymbol{v}(t,x,v), \\ \dot{\boldsymbol{v}}(t,x,v) = E(t,\boldsymbol{x}(t,x,v)) + \frac{\boldsymbol{x}(t,x,v)-\xi(t)}{|\boldsymbol{x}(t,x,v)-\xi(t)|^3}, \quad (\boldsymbol{x},\boldsymbol{v})(0,x,v) = (x,v). \end{cases} \tag{5.3.4}$$

(ii) For all $t \in \mathbb{R}_+$, the map $(x,v) \mapsto (\boldsymbol{x}(t,x,v), \boldsymbol{v}(t,x,v)$ preserves the Lebesgue measure on $\mathbb{R}^3 \times \mathbb{R}^3$.

Since the solution constructed in Theorem 6 has bounded moments of order higher than 6, the field E belongs to $L^\infty_{\text{loc}}(\mathbb{R}_+, C^{0,\alpha}(\mathbb{R}^3))$ for some $0 < \alpha < 1$ (see Corollary 2 in [7]). It is actually also continuous in time. Therefore given such a field E, for all (ξ_0, η_0) and for all $(x \neq \xi_0, v)$ the corresponding ODE (5.3.2) and (5.3.4) have at least one solution, which is C^1 in time, as long as there are no collisions between the plasma trajectories and the charge. We shall see that the repulsive nature of the interaction between the plasma and the charge prevents collisions in finite time to occur, so that the flow $(\boldsymbol{x}, \boldsymbol{v})$ is globally defined.
We stress that, since E is only Hölder continuous, uniqueness of the solution to (5.3.4) (or (5.3.3)) does not hold a priori for all initial condition (x,v) (or (ξ_0,η_0)). However we mention that, according to the previous works by Hauray [6] and, e.g., Ambrosio and Crippa ([1], Theorem 19), such flow $(\boldsymbol{x}, \boldsymbol{v})$ indeed corresponds to the notion of generalized flow à la Di

Perna and Lions and is unique among the maps on $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfying (i) and the non-concentration property (ii). The remainder of this Section is organized as follows. The next section is devoted to the proof of Theorem 6. The general procedure, which follows the lines of [7], consists in deriving a priori estimates for the moments for a sequence of smooth solutions to (5.3.1)–(5.3.3) obtained by regularizing the initial density in order to obtain a global solution by compactness arguments.

In Subsection 5.3.3 we gather some basic facts and a priori estimates for the modified Vlasov-Poisson system (5.3.1). We also derive some first estimates for the energy moments. In Subsection 5.3.4 we introduce a notion of almost-free flow, which enables to express the solution of (5.3.1) by means of Duhamel's formula with a suitable source term. Then, in Subsection 5.3.5 we establish intermediate a priori estimates for the moments, which as a byproduct ensure that the moments are uniformly bounded for small times. These estimates are exploited to show that the moments are uniformly bounded for all times in Subsection 5.3.6. They eventually provide a global solution satisfying the assumptions of Theorem 6. An appendix is also devoted to the proof of technical estimates on the almost-free flow.

5.3.2 Some useful interpolation estimates

Before studying the dynamics of the Vlasov-Poisson system, we recall a collection of well-known interpolation inequalities that we shall apply later to the solutions of (5.3.1)–(5.3.3). All of them may be found in [7].

Proposition 3. *Let $f = f(x, v) \geq 0$. Let $b > a \geq 0$. Then for all $x \in \mathbb{R}^3$*

$$\int_{\mathbb{R}^3} |v|^a f(x,v)\,dv \leq C\|f\|_{L^\infty}^{\frac{3+a}{3+b}} \left(\int |v|^b f(x,v)\,dv\right)^{\frac{3+a}{3+b}} \tag{5.3.5}$$

with C a numerical constant. In particular, setting $\rho(x) = \int_{\mathbb{R}^3} f(x,v)\,dv$ we have

$$\|\rho\|_{L^{\frac{b+3}{3}}} \leq C\|f\|_{L^\infty}^{\frac{3}{3+b}} \left(\int\int |v|^b f(x,v)\,dx\,dv\right)^{\frac{3}{3+b}}. \tag{5.3.6}$$

Proof. For all $R \geq 0$

$$\int |v|^a f(x,v)\,dv \leq R^{a-b} \int |v|^b f(x,v)\,dv + CR^{3+a}\|f\|_\infty, \tag{5.3.7}$$

and the estimate (5.3.5) is obtained by optimizing $R > 0$, cf. also the proof of estimate (14) in [7]. Setting $a = 0$ we obtain (5.3.6). $\square$

Proposition 4. *Let $f \geq 0$ be in $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, such that $\rho(x) = \int_{\mathbb{R}^3} f(x,v)\,dv \in L^s(\mathbb{R}^3)$ (for some $s \in [1,\infty]$) and $E = \rho * (x \mapsto x/|x|^3)$. Then for $s \in]1,3[$,*

$$\|E\|_{L^{\frac{3s}{3-s}}} \leq C\,\|\rho\|_{L^s}, \tag{5.3.8}$$

and for $s > 3$,

$$\|E\|_{L^\infty} \leq C\|\rho\|_{L^s}. \tag{5.3.9}$$

Proof. The inequalities are direct consequences of Sobolev inequalities and the fact that $E = 4\pi \nabla_x \Delta_x^{-1} \rho$. $\square$

5.3.3 Some first estimates on the growth of the moments

We now turn to the study of the system (5.3.1)–(5.3.3). In the remainder of this article, we fix $T > 0$.

In the sequel, we call *classical solution* any solution (f, ξ) of (5.3.1)–(5.3.3) on $[0, T]$, with initial condition (f_0, ξ_0, η_0) satisfying the assumptions in Theorem 6, such that *moreover* f_0 is C^1, compactly supported, and vanishes in a neighborhood of ξ_0, which satisfies $f \in C_c^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$, $\xi \in C^2([0, T])$, and such f is transported by the classical flow $(\boldsymbol{x}, \boldsymbol{v})$ of (5.3.4). The existence (and uniqueness) of classical solutions corresponding to such initial data is ensured by [12]. Our purpose is to establish relevant a priori estimates for (f, ξ) on $[0, T]$, which will eventually lead to the existence of a solution to (5.3.1)–(5.3.3) by standard compactness arguments. As already mentioned, such a priori estimates concern the moments of order $m < m_0$, which are defined in Definition 1 below.

We start with a few basic properties of the Vlasov-Poisson system.

Proposition 5. *Let (f, ξ) be a classical, compactly supported solution of* (5.3.1)-(5.3.3) *on $[0, T]$.*
Then, the norms

$$\|f(t)\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}, \quad 1 \le p \le \infty,$$

and the energy

$$\mathcal{H}(t) = \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f(t, x, v)\, dv\, dx + \frac{1}{2} |\eta(t)|^2 + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho(t, x)\, \rho(t, y)}{|x - y|}\, dx dy + \int_{\mathbb{R}^3} \frac{\rho(t, x)}{|x - \xi(t)|}\, dx$$

are conserved in time. In particular, the mass

$$\mathcal{M}(t) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v)\, dx\, dv$$

is conserved in time.

Proof. The conservation of the L^p norms is an immediate consequence of the fact that f is transported by a Lebesgue measure-preserving flow.

We only detail the computation of the energy conservation estimate:

$$
\begin{aligned}
&\frac{d}{dt}\Big\{ \iint f\,\frac{|v|^2}{2}\,dx\,dv + \frac{1}{2}|\eta|^2 + \frac{1}{2}\iint \frac{\rho(x)\,\rho(y)}{|x-y|}\,dx\,dy + \int \frac{\rho(x)}{|x-\xi|}\,dx \Big\} \\
&= \int\int v\cdot(E+F)\,f\,dvdx + \eta\cdot E(\xi) \\
&- \iint \frac{\nabla_x\cdot\int v\,f\,dv}{|x-y|}\,\rho(y)\,dxdy - \int \frac{\nabla_x\cdot\int v\,f\,dv}{|x-\xi|}\,dx \\
&- \int \rho(x)\,\eta\cdot\frac{\xi-x}{|\xi-x|^3}\,dx \\
&= 0.
\end{aligned}
$$

□

For the initial data $(f_0,(\xi_0,\eta_0))$ considered in the setting of Theorem 6, the energy is initially finite; indeed Proposition 3 yields $\rho_0\in L^1\cap L^{5/3}$ therefore $\iint \rho(x)\rho(y)/|x-y|\,dx\,dy$ is finite by Hölder estimates; on the other hand the other terms are clearly finite by assumption (ii). So we immediately get the

Proposition 6. *Under the same assumptions on (f,ξ) as in proposition 5, we have*

$$\sup_{t\in[0,T]} |\eta(t)| \le \sqrt{2\mathcal{H}(0)} \tag{5.3.10}$$

and

$$\sup_{t\in[0,T]} |\xi(t)| \le |\xi_0| + \sqrt{2\mathcal{H}(0)}\,T. \tag{5.3.11}$$

We may assume that $R_0 = |\xi_0| + \sqrt{2\mathcal{H}(0)} > 2$.

Proof. The first inequality is a consequence of the conservation of the energy. The second one comes out of the integration w.r.t. time of the first one. □

Another well-known consequence of the conservation of the energy is the following

Proposition 7. *Under the same assumptions on (f,ξ) as in Proposition 5, we have*

$$\sup_{t\in[0,T]} \|\rho(t)\|_{L^{5/3}} \le C,$$

and for all $\frac{3}{2} < r \le \frac{15}{4}$,

$$\sup_{t\in[0,T]} \|E(t)\|_{L^r} \le C,$$

with C a constant depending only on $\mathcal{H}(0)$ and $\|f_0\|_\infty$ and r.

Proof. The first estimate is a consequence of (5.3.6) with $b=2$ and the fact that the moment of order 2 is controlled by the energy. The second estimate is deduced from the first one and (5.3.8), using the fact that $\rho\in L^\infty([0,T],L^1\cap L^{5/3}(\mathbb{R}^3))$. □

We now give our definition of energy moments.

Definition 1. We define the energy function

$$h(t,x,v) = \frac{|v-\eta(t)|^2}{2} + \frac{1}{|x-\xi(t)|} + K, \tag{5.3.12}$$

where $K \geq 1$ is a constant sufficiently large with respect to $\mathcal{H}(0)$ (for example $K = \mathcal{H}(0)+1$). In view of Proposition 6 one can choose K in such a way that

$$|v| \leq 3\sqrt{h}(t,x,v) \quad \forall (t,x,v) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3.$$

Then, we set, for $k \in \mathbb{R}_+$,

$$\tilde{H}_k(t) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} h(t,x,v)^{k/2} f(t,x,v) dx\, dv \tag{5.3.13}$$

and

$$H_k(t) = \sup_{s\in[0,t]} \tilde{H}_k(s) = \sup_{s\in[0,t]} \iint_{\mathbb{R}^3\times\mathbb{R}^3} h(s,x,v)^{k/2} f(s,x,v) dx\, dv. \tag{5.3.14}$$

A first basic observation is that the energy moments H_k control the velocity moments M_k defined in [7], namely

$$M_k(t) = \sup_{s\in[0,t]} \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f(s,x,v) dx\, dv \leq 3^k H_k(t). \tag{5.3.15}$$

Notation. In all the following, the notation C will refer to a constant depending only on the quantities $\mathcal{H}(0)$, $\mathcal{M}_0$, $\|f_0\|_\infty$, ξ_0, $H_m(0)$, for $m < m_0$, and T. Note that in the assumptions of Theorem 6 these quantities are finite (in the process of approximation leading to the existence, they will be bounded with respect to the regularization parameter).

Lemma 9. *Let (f,ξ) be a classical, compactly supported solution of* (5.3.1)-(5.3.3) *on* $[0,T]$. *We have for all $t \in [0,T]$ and for $k \in \mathbb{R}_+$*

$$\frac{d}{dt}\tilde{H}_k(t) \leq C\Big(\|E(t)\|_{L^{k+3}} + |E(t,\xi(t))|\Big) H_k(t)^{\frac{k+2}{k+3}} \tag{5.3.16}$$

and therefore,

$$H_k(t) \leq C \left\{ H_k(0) + \left(\int_0^t \Big\{ \|E(s)\|_{L^{k+3}} + |E(s,\xi(s))| \Big\} ds \right)^{k+3} \right\}. \tag{5.3.17}$$

Proof. Since $f \in C_c^1([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$ is a classical solution of (5.3.1), we may compute

$$\begin{aligned}\frac{d}{dt}\tilde{H}_k(t) &= \frac{k}{2}\int h^{k/2-1} f \left\{\partial_t h + v\cdot\nabla_x h + (E+F)\cdot\nabla_v h\right\}(t,x,v)\, dx\, dv \\ &= \frac{k}{2}\int h^{k/2-1} f \left\{(v-\eta(t))\cdot\big(E(t,x) - E(t\xi(t))\big)\right\} dx\, dv.\end{aligned}$$

We remark that the choice of the energy function h enabled to get rid of the singular field in the second equality.
Therefore

$$\frac{d}{dt}\tilde{H}_k(t) \leq C \int |E(t,x)| h^{(k-1)/2} f(t,x,v)\,dx\,dv + |E(t,\xi)| \int h^{(k-1)/2} f(t,x,v)\,dx\,dv. \quad (5.3.18)$$

In order to bound the first term of the right-hand side in (5.3.18) we use interpolation arguments from [7] that we recall here for sake of clarity. First, we have thanks to Hölder inequality

$$\int |E(t,x)| h^{(k-1)/2} f(t,x,v)\,dx\,dv \leq C \|E(t)\|_{L^{k+3}} \left\| \int h^{(k-1)/2} f(t,\cdot,v)\,dv \right\|_{L^{\frac{k+3}{k+2}}}.$$

Next, we have for $x \in \mathbb{R}^3$ and for $R > 0$,

$$\begin{aligned}
\int h^{(k-1)/2} f(t,x,v)\,dv &= \int_{h^{1/2} \leq R} h^{(k-1)/2} f(t,x,v)\,dv + \int_{h^{1/2} \geq R} h^{(k-1)/2} f(t,x,v)\,dv \\
&\leq R^{k-1} \int_{|v| \leq CR} f(t,x,v)\,dv + R^{-1} \int_{h^{1/2} \geq R} h^{k/2} f(t,x,v)\,dv \\
&\leq C \|f(t)\|_{L^\infty} R^{k+2} + R^{-1} \int h^{k/2} f(t,x,v)\,dv.
\end{aligned}$$

We have used the fact that $|v| \leq C h^{1/2}$ in the second inequality. Now, optimizing w.r.t. R, and using that $\|f(t)\|_{L^\infty} = \|f_0\|_{L^\infty}$ we find

$$\int h^{(k-1)/2} f(t,x,v)\,dv \leq C \left(\int h^{k/2} f(t,x,v)\,dv \right)^{(k+2)/(k+3)}.$$

So finally, integrating in x, we obtain

$$\left\| \int h^{(k-1)/2} f(t,\cdot,v)\,dv \right\|_{L^{\frac{k+3}{k+2}}} \leq C H_k(t)^{(k+2)/(k+3)},$$

and we are led to

$$\int\int |E(t,x)| h^{(k-1)/2} f(t,x,v)\,dx\,dv \leq C \|E(t)\|_{L^{k+3}}\, H_k(t)^{(k+2)/(k+3)}. \quad (5.3.19)$$

We next estimate the second term in (5.3.18). Applying again Hölder inequality yields

$$\int\int h^{(k-1)/2}\, f(t,x,v)\,dx\,dv \leq \left(\int\int f(t,x,v)\,dx\,dv \right)^{1/k} \left(\int\int h^{k/2}\, f(t,x,v)\,dx\,dv \right)^{(k-1)/k}$$

so that, since $\mathcal{M}(t) = \mathcal{M}_0$,

$$\int\int h^{(k-1)/2} f(t,x,v)\,dx\,dv \leq C H_k(t)^{(k-1)/k}.$$

Since $(k-1)/k \leq (k+2)/(k+3)$ and $H_k(t) \geq 1$, it follows that

$$|E(t,\xi)| \int h^{(k-1)/2} f(t,x,v)\,dx\,dv \leq C |E(t,\xi)| H_k(t)^{(k+2)/(k+3)}. \quad (5.3.20)$$

Gathering estimates (5.3.19) and (5.3.20) we are led to the conclusion of Lemma 9. □

In order to exploit Lemma 9, we now need to control the electric fields $|E(\xi)|$ and $\|E\|_{L^{k+3}}$. Applying estimate (5.3.30) in Proposition 10 we readily get $|E(\xi)| \leq CH_k^{3/(k+3)}$ if $k > 6$, but this inequality is too rough to provide an estimate for $H_k(t)$ by means of the estimates in Lemma 9. In fact, when $\mathcal{M}_0 < 1$ one can improve the previous estimate on the electric field computed at the point charge by the following virial-type argument.

Proposition 8. *Under the assumptions on (f, ξ) as in Proposition 5, we have*

$$\int_0^t |E(s, \xi(s))| \, ds \leq C.$$

Remark 8. *This is the only point of the proof of Theorem 6 in which we use the assumption (i).*

Proof. Let $(\boldsymbol{x}(s), \boldsymbol{v}(s)) = (\boldsymbol{x}(s, x, v), \boldsymbol{v}(s, x, v))$ be a plasma trajectory on $[0, T]$. Using the system of ODE (5.3.2) and (5.3.4), we compute

$$\begin{aligned}\frac{d^2}{ds^2}|\boldsymbol{x}(s) - \xi(s)| &= \frac{|\boldsymbol{v}(s) - \eta(s)|^2}{|\boldsymbol{v}(s) - \xi(s)|} + \frac{1}{|\boldsymbol{x}(s) - \xi(s)|^2} \\ &\quad + \frac{\big(\boldsymbol{x}(s) - \xi(s)\big) \cdot \big(E(s, \boldsymbol{x}(s)) - E(s, \xi(s))\big)}{|\boldsymbol{x}(s) - \xi(s)|} - \frac{[(\boldsymbol{x}(s) - \xi(s)) \cdot (\boldsymbol{v}(s) - \eta(s))]^2}{|\boldsymbol{x}(s) - \xi(s)|^3}.\end{aligned}$$

Therefore

$$\frac{1}{|\boldsymbol{x}(s) - \xi(s)|^2} \leq \frac{d^2}{ds^2}|\boldsymbol{x}(s) - \xi(s)| + |E(s, \boldsymbol{x}(s))| + |E(s, \xi(s))|. \tag{5.3.21}$$

On the other hand, since f is transported by the measure-preserving flow $(\boldsymbol{x}, \boldsymbol{v})$, we have by changing variable

$$|E(s, \xi(s))| \leq \iint \frac{f(s, x, v)}{|x - \xi(s)|^2} \, dx \, dv = \iint \frac{f_0(x, v)}{|\boldsymbol{x}(s, x, v) - \xi(s)|^2} \, dx \, dv.$$

Therefore inserting (5.3.21) we get

$$\begin{aligned}\int_0^t |E(s, \xi(s))| \, ds &\leq \iint f_0(x, v) \left(\int_0^t \frac{d^2}{ds^2}|\boldsymbol{x}(s) - \xi(s)| \, ds \right) dx \, dv \\ &\quad + \int_0^t \left(\iint f_0(x, v) |E(s, \boldsymbol{x}(s, x, v))| \, dx \, dv \right) ds + \mathcal{M}_0 \int_0^t |E(s, \xi(s))| \, ds.\end{aligned} \tag{5.3.22}$$

For the first term in the right-hand side of (5.3.22), we have

$$\begin{aligned}\iint f_0(x, v) \left(\int_0^t \frac{d^2}{ds^2}|\boldsymbol{x}(s) - \xi(s)| \, ds \right) dx \, dv &= \iint f_0(x, v) \left[\frac{d}{ds}|\boldsymbol{x} - \xi| \right]_{s=0}^{s=t} dx \, dv \\ &\leq \iint f_0(x, v) \left(\left| \frac{d}{ds}|\boldsymbol{x} - \xi| \right|(t) + \left| \frac{d}{ds}|\boldsymbol{x} - \xi| \right|(0) \right) dx \, dv \\ &\leq 2 \sup_{t \in [0, T]} \iint f_0(x, v) |\boldsymbol{v}(t, x, v) - \eta(t)| \, dx \, dv \\ &= 2 \sup_{t \in [0, T]} \iint f(t, x, v) |v - \eta(t)| \, dx \, dv.\end{aligned}$$

Hence, by Hölder's inequality, we obtain

$$\iint f_0(x,v)\left(\int_0^t \frac{d^2}{ds^2}|X(s)-\xi(s)|\,ds\right)dx\,dv \le C \sup_{t\in[0,T]} \mathcal{M}(t)^{1/2}\mathcal{H}(t)^{1/2} \le C.$$

We turn to the second term in (5.3.22). We have by changing variable backwards

$$\begin{aligned}\int_0^t \left(\iint f_0(x,v)|E(s,X(s,x,v))|\,dx\,dv\right)ds &= \int_0^t \left(\iint f(s,x,v)|E(s,x)|\,dx\,dv\right)ds\\ &= \int_0^t \left(\int \rho(s,x)|E(s,x)|\,dx\right)ds\\ &\le C\int_0^t \|\rho(s)\|_{L^{5/3}}\|E(s)\|_{L^{5/2}}\,ds \le C.\end{aligned}$$

We used Proposition 7 in the last inequality.
Therefore coming back to (5.3.22), we find

$$\int_0^t |E(s,\xi(s))|\,ds \le C + \mathcal{M}_0\int_0^t |E(s,\xi(s))|\,ds.$$

The conclusion of Proposition 8 follows from the assumption (i) on $\mathcal{M}_0$. □

5.3.4 The modified flow and the Duhamel formula

Let (f,ξ) be a classical, compactly supported solution of (5.3.1)-(5.3.3) on $[0,T]$. We decompose the electric field and the force field into two parts:

$$E = E_{\text{int}} + E_{\text{ext}}, \qquad F = F_{\text{int}} + F_{\text{ext}},$$

where

$$E_{\text{int}} = \left(x \mapsto \chi_R(x)\,\frac{x}{|x|^3}\right) * \rho, \qquad F_{\text{int}}(t,x) = F(t,x)\chi_R(x-\xi(t)),$$

χ_R being a smooth cutoff function such that $\chi_R(x)=1$ on $B(0,R)$, $\chi_R(x)=0$ on $B(0,2R)^c$ and $0<\chi_R(x)<1$ on $\mathbb{R}^3$, and where $R>1$ is large and will be determined later in terms of $\mathcal{H}(0)$, $\mathcal{M}_0$, $\|f_0\|_\infty$, ξ_0, $H_m(0)$, for $m<m_0$, and T.
We have

$$\|E_{\text{ext}}\|_{L^\infty} + \|\nabla E_{\text{ext}}\|_{L^\infty} + \|D^2E_{\text{ext}}\|_{L^\infty} \le \frac{\|\rho_0\|_{L^1}}{R^2} \tag{5.3.23}$$

and

$$\|F_{\text{ext}}\|_{L^\infty} + \|\nabla F_{\text{ext}}\|_{L^\infty} + \|D^2F_{\text{ext}}\|_{L^\infty} \le \frac{1}{R^2}. \tag{5.3.24}$$

As in [7] we write the Vlasov equation using the internal part of E and F as a source term:

$$\partial_t f + v\cdot\nabla_x f + (E_{\text{ext}}+F_{\text{ext}})\cdot\nabla_v f = -(E_{\text{int}}+F_{\text{int}})\cdot\nabla_v f. \tag{5.3.25}$$

The reason why we do not consider the free transport (namely we do not consider the full field as a source term) will appear Subsection 5.3.6.

Next, we fix $t > 0$ and we define the flow map $(x, v) \mapsto (X, V)(x, v)$ such that

$$\begin{cases} \dot{X}(s) = -V(s), & X(0) = x, \\ \dot{V}(s) = -(E_{ext} + F_{ext})(t-s, X(s)), & V(0) = v, \quad 0 \le s \le t. \end{cases} \tag{5.3.26}$$

This is the backward flow, not to be merged with the inverse flow of $(\boldsymbol{x}, \boldsymbol{v})$. It preserves the Lebesgue's measure on $\mathbb{R}^3 \times \mathbb{R}^3$. Note that if the external field vanished we would obtain the free flow $X(s, x, v) = x - vs, V(s, x, v) = v$, and if we considered the total field in (5.3.26) we would obtain the inverse of $(\boldsymbol{x}, \boldsymbol{v})$. We shall sometimes write $(X(s), V(s))$ instead of $(X(s, x, v), V(s, x, v))$.

Using the invertibility properties of the flow listed in the appendix one can establish the analog of Proposition 3:

Proposition 9. *Let (f, ξ) be a classical, compactly supported solution of* (5.3.1)-(5.3.3) *on $[0, T]$. Let $0 \le s, \tau \le T$. Let $b > a \ge 0$. Then*

$$\int_{\mathbb{R}^3} |v|^a f(\tau, X(s,x,v), V(s,x,v))\, dv \le C \|f_0\|_{L^\infty}^{\frac{3+a}{3+b}} \left(\int |v|^b f(\tau, X(s,x,v), V(s,x,v))\, dv \right)^{\frac{3+a}{3+b}}. \tag{5.3.27}$$

In particular, setting $\tilde{\rho}(x) = \int_{\mathbb{R}^3} f(\tau, X(s,x,v), V(s,x,v)\, dv$ we have

$$\|\tilde{\rho}\|_{L^{\frac{b+3}{3}}} \le C \|f_0\|_{L^\infty}^{\frac{3}{3+b}} \left(1 + \int \int |v|^b\, f(\tau, x, v)\, dx\, dv \right)^{\frac{3}{3+b}} \tag{5.3.28}$$

with C a numerical constant depending on T.

Proof. For the first inequality this is exactly the same proof as for Proposition 3, estimate (5.3.5). The second inequality is obtained thanks to the bound $|V(s,x,v) - v| \le CT$ (see (5.4.12)), and using the fact that $(X(s), V(s))$ preserves Lebesgue's measure. Recall also that $\|f(\tau)\|_{L^\infty} = \|f_0\|_{L^\infty}$. □

A consequence of Propositions 4 and 9 is

Proposition 10. *Let (f, ξ) be a classical, compactly supported solution of* (5.3.1)-(5.3.3) *on $[0, T]$. Let $0 \le s, \tau \le T$. Let $\tilde{\rho}(x) = \int_{\mathbb{R}^3} f(\tau, X(s,x,v), V(s,x,v)\, dv$ and $\tilde{E} = \tilde{\rho} * (x \mapsto x/|x|^3)$. If $m \in]3, 6[$ we have*

$$\|\tilde{E}\|_{L^{\frac{3(m+3)}{6-m}}} \le C \left(1 + \iint |v|^m f(\tau, x, v)\, dx\, dv \right)^{\frac{3}{m+3}} \le C H_m(\tau)^{\frac{3}{m+3}}, \tag{5.3.29}$$

and if $m > 6$,

$$\|\tilde{E}\|_{L^\infty} \le C \left(1 + \iint |v|^m f(\tau, x, v)\, dx\, dv \right)^{\frac{3}{m+3}} \le C H_m(\tau)^{\frac{3}{m+3}} \tag{5.3.30}$$

where $C > 0$ is a numerical constant which depends on $\|f\|_{L^\infty}$ and T.

Proof. We apply (5.3.8) and (5.3.9) with $s = (m+3)/3$, and conclude thanks to (5.3.28). $\square$

Using Duhamel formula we can express the solution of (5.3.25) as follows:

$$f(t,x,v) = f_0(X(t,x,v), V(t,x,v)) + \int_0^t \big(\nabla_v \cdot [(E_{\text{int}} + F_{\text{int}})\, f]\big)(t-s, X(s,x,v), V(s,x,v))\, ds. \tag{5.3.31}$$

Proposition 11. *Under the assumptions on (f,ξ) as in Proposition 5, we have for $m \geq 3$*

$$\|E(t)\|_{L^{m+3}} \leq C + C \left\| \int_0^t s \int_{\mathbb{R}^3} (E_{int} + F_{int}) f(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}}. \tag{5.3.32}$$

Proof. By (5.3.31) we have

$$\begin{aligned} \rho(t,x) &= \int_{\mathbb{R}^3} f_0(X(t), V(t))\, dv + \int_0^t \int \big(\nabla_v \cdot [(E_{\text{int}} + F_{\text{int}})\, f]\big)(t-s, X(s), V(s))\, dv\, ds \\ &= \rho_1(t,x) + \rho_2(t,x). \end{aligned} \tag{5.3.33}$$

Let us set $E_1 = \rho_1 * x/|x|^3$. By Proposition 10 we have by interpolation, since $3(m+3)/(6-m) \geq m+3$ and since $H_m(0)$ is finite,

$$\|E_1\|_{m+3} \leq C.$$

For the term ρ_2 and the corresponding field E_2 we have to work more and use the appendix (properties of the flow). $\square$

Next sections are devoted to the control of the right-hand of (5.3.32).

5.3.5 Intermediate small time estimates for the moments

The purpose of this paragraph is to establish uniform estimates for the moments on $[0,T]$.

Proposition 12. *Under the assumptions on (f,ξ) as in Proposition 5, let $t \in [0, \min(1,T)]$ and let $m \geq 3$.*
Let $0 < \gamma < 1$. We introduce k defined by $k+3 = (m+3)(1+\gamma)$ (note that $k > m$), and we define δ by $\delta = \frac{\gamma}{1+(\gamma+1)(m+3)} \in]0,1[$.
Then we have

$$\left\| \int_0^t s \int_{\mathbb{R}^3} [(E_{int} + F_{int}) f](t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}} \leq C(\gamma, m)\, t^\delta\, H_k(t)^{\frac{1}{m+3}}.$$

Remark 9. *Taking γ small in Proposition 12, we realize that $k > m$ may be chosen as close as we want to m, therefore the estimate $\|E(t)\|_{L^{m+3}} \leq C H_m(t)^{1/(m+3)}$, which in view of Lemma 9 would be enough to obtain an estimate on $H_m(t)$, is close to be achieved.*

Proof. By Proposition 7, the field E_{int} belongs to $L^\infty([0,T], L^{r_1}(\mathbb{R}^3))$ for all $3/2 < r_1 \leq 15/4$. Since this is the electric field produced by the bounded density f, we can use the estimates of [7] as a blackbox : more precisely by estimates (31)-(32) and (28')-(40) of [7] we get for all $3/2 < r_1 \leq 15/4$

$$\left\| \int_0^t s \int_{\mathbb{R}^3} (E_{\text{int}} f)(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}} \leq C(r_1, m)\, t^{2-\frac{3}{r_1}}\, M_{k_1}(t)^{\frac{1}{m+3}},$$

where $k_1 > m$ is defined by $k_1 + 3 = (m+3)(3 - 3/r_1)$, which by (5.3.15) yields

$$\left\| \int_0^t s \int_{\mathbb{R}^3} (E_{\text{int}} f)(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}} \leq C(r_1, m)\, t^{2-\frac{3}{r_1}}\, H_{k_1}(t)^{\frac{1}{m+3}}. \tag{5.3.34}$$

We then introduce

$$I(x) = \int_0^t s \int_{\mathbb{R}^3} (F_{\text{int}} f)(t-s, X(s), V(s))\, dv\, ds.$$

In the following we will write ξ instead of $\xi(t-s)$ when not misleading.

Step 1. Local estimate for I

We recall that by Proposition 6 there exists $R_0 \geq 4$ such that $\sup_{t\in[0,T]} |\xi(t)| \leq R_0$. We set $B = B(0, 3R)$. We take $R > R_0$.

Let $0 < \varepsilon < 2/(m+3)$ be a small parameter and let us pick $3/(2+\varepsilon) < r_2 < 3/2$. By Hölder inequality we get

$$\begin{aligned}
&\|I\|_{L^{m+3}(B)} \\
&= \left\| \int_0^t s^{1+\varepsilon} \int \frac{|F_{\text{int}}|(t-s, X(s))}{|X(s)-x|^\varepsilon} \left(\frac{|X(s)-x|}{s}\right)^\varepsilon f(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}(B)} \\
&\leq \|f_0\|_{L^\infty}^{1-\frac{1}{r_2'}} \left\| \int_0^t s^{1+\varepsilon} \left(\int \frac{|F_{\text{int}}|^{r_2}(t-s, X(s))}{|X(s)-x|^{\varepsilon r_2}} dv \right)^{\frac{1}{r_2}} \left(\int \left(\frac{|X(s)-x|}{s}\right)^{\varepsilon r_2'} f(t-s, X(s), V(s)) dv \right)^{\frac{1}{r_2'}} ds \right\|_{L^{m+3}(B)}
\end{aligned}$$

For all fixed $x \in B$ and $0 \leq s \leq T$ we perform the change of variable

$$y = x - X(s, x, v).$$

Then by (5.4.11) $|y| \leq s(|v|+1)$ if R large enough (depending on T). Moreover we have

$$\int \frac{|F_{\text{int}}|^{r_2}(t-s, X(s))}{|X(s)-x|^{\varepsilon r_2}}\, dv = \int \frac{|F_{\text{int}}|^{r_2}(t-s, x-y)}{|y|^{\varepsilon r_2}} \det(\nabla_v X(s))|^{-1}\, dy$$

and, according to the estimates in the Appendix (see (5.4.9)) we have $|\det(\nabla_v X(s))|^{-1} \leq C/s^3$.

So we obtain

$$
\begin{aligned}
&\|I\|_{L^{m+3}(B)} \\
&\leq C \left\| \int_0^t \ s^{1-\frac{3}{r_2}+\varepsilon} \ \left(\int \frac{|F_{\mathrm{int}}|^{r_2}(t-s,x-y)}{|y|^{\varepsilon r_2}} \, dy \right)^{\frac{1}{r_2}} \left(\int (1+|v|^{\varepsilon r_2'}) f(t-s,X(s),V(s)) \, dv \right)^{\frac{1}{r_2'}} \ ds \right\|_{L^{m+3}(B)} \\
&\leq C \int_0^t \ s^{1-\frac{3}{r_2}+\varepsilon} \Big\{ \int_{|x|\leq 3R} \ \left(\int_{|x-\xi-y|\leq R} \frac{dy}{|y|^{\varepsilon r_2} |x-\xi-y|^{2r_2}} \right)^{\frac{m+3}{r_2}} \\
&\qquad\qquad \left(\int (1+|v|^{\varepsilon r_2'}) f(t-s,X(s),V(s)) \, dv \right)^{\frac{(m+3)}{r_2'}} \ dx \Big\}^{\frac{1}{m+3}} \, ds \\
&= C \int_0^t s^{1-\frac{3}{r_2}+\varepsilon} \ \left\{ \int_{|x|\leq 3R} \ |x-\xi|^{(\frac{3}{r_2}-2-\varepsilon)(m+3)} \left(\int (1+|v|^{\varepsilon r_2'}) f(t-s,X(s),V(s)) \, dv \right)^{\frac{(m+3)}{r_2'}} \ dx \right\}^{\frac{1}{m+3}} \ ds.
\end{aligned}
$$

We now set

$$r_2 = \frac{3}{2+\varepsilon/2}$$

and we define

$$p = \frac{2}{\varepsilon\,(m+3)}.$$

Note that $p>1$ and

$$\frac{3}{2+\frac{1}{m+3}} < r_2 < \frac{3}{2}. \tag{5.3.35}$$

Moreover

$$-(\frac{3}{r_2}-2-\varepsilon)(m+3)p = \frac{\varepsilon}{2}(m+3)p = 1 < 3.$$

Applying Hölder's inequality, we obtain

$$
\begin{aligned}
&\|I\|_{L^{m+3}(B)} \\
&\leq C \int_0^t s^{-1+\frac{\varepsilon}{2}} \ \left(\int_{|x|\leq 3R} |x-\xi|^{-\frac{\varepsilon}{2}(m+3)p} \, dx \right)^{\frac{1}{(m+3)p}} \times \\
&\quad \times \left(\int_{|x|\leq 3R} \left(\int (1+|v|^{\varepsilon r_2'}) f(t-s,X(s),V(s)) \, dv \right)^{\frac{(m+3)}{r_2'}p'} \ dx \right)^{\frac{1}{(m+3)p'}} \ ds \\
&\leq C t^{\frac{\varepsilon}{2}} \sup_{\tau,s\in[0,t]} \left\{ \int \left(\int (1+|v|^{\varepsilon r_2'}) f(\tau,X(s),V(s)) \, dv \right)^{\frac{(m+3)}{r_2'}p'} \ dx \right\}^{\frac{1}{(m+3)p'}}.
\end{aligned}
$$

We now focus on the right-hand side

$$\sup_{\tau,s\in[0,t]} \left\{ \int \left(\int (1+|v|^{\varepsilon r_2'}) f(\tau,X(s),V(s)) \, dv \right)^{\frac{(m+3)}{r_2'}p'} \ dx \right\}^{\frac{1}{(m+3)p'}}. \tag{5.3.36}$$

Let us introduce k_2 such that

$$\left(\frac{3+\varepsilon r_2'}{3+k_2}\right)\left(\frac{m+3}{r_2'}p'\right)=1$$

and apply (5.3.27) and (5.3.28) with this choice and $a=\varepsilon r_2'$, $b=k_2$. Note that $b>a$ since

$$\frac{(m+3)}{r_2'}p'>1 \iff (m+3)(1/3-\varepsilon/6)>1-\varepsilon\,(m+3)/2 \iff 1/3>1/(m+3)-\varepsilon/3,$$

which holds as soon as $m>0$ (remember that $m>3$ in this proposition).
We obtain

$$\sup_{\tau,s\in[0,t]}\left\{\int\left(\int |v|^{\varepsilon r_2'}f(\tau,X(s),V(s))\,dv\right)^{\frac{(m+3)}{r_2'}p'}dx\right\}^{\frac{1}{(m+3)p'}}\leq$$

$$\leq C\sup_{\tau,s\in[0,t]}\left(\iint |v|^{k_2}f(\tau,X(s),V(s))\,dx\,dv\right)^{\frac{1}{(m+3)p'}}.$$

Similarly, we introduce k_2' such that

$$\left(\frac{3}{3+k_2'}\right)\left(\frac{m+3}{r_2'}p'\right)=1$$

and apply (5.3.27) and (5.3.28) with this choice and $a=0$, $b=k_2'>a$. Since $k_2'<k_2$ we obtain

$$\sup_{\tau,s\in[0,t]}\left\{\int\left(\int f(\tau,X(s),V(s))\,dv\right)^{\frac{(m+3)}{r_2'}p'}dx\right\}^{\frac{1}{(m+3)p'}}\leq$$

$$\leq C\sup_{\tau,s\in[0,t]}\left(\iint |v|^{k_2'}f(\tau,X(s),V(s))\,dx\,dv\right)^{\frac{1}{(m+3)p'}}\leq$$

$$\leq C\sup_{\tau,s\in[0,t]}\left(\iint (1+|v|^{k_2})f(\tau,X(s),V(s))\,dx\,dv\right)^{\frac{1}{(m+3)p'}}\leq$$

$$\leq C+C\sup_{\tau,s\in[0,t]}\left(\iint |v|^{k_2}f(\tau,X(s),V(s))\,dx\,dv\right)^{\frac{1}{(m+3)p'}}.$$

Finally, we obtain

$$\|I\|_{L^{m+3}(B)}\leq Ct^{\frac{\varepsilon}{2}}\,H_{k_2}(t)^{\frac{1}{(m+3)p'}}.$$

Since $p'\geq 1$, making explicit the dependence of the constants, we get for any r_2 satisfying the condition (5.3.35)

$$\|I\|_{L^{m+3}(B)}\leq C(r_2,m)\,t^{\frac{3}{r_2}-2}\,H_{k_2}(t)^{\frac{1}{m+3}}, \tag{5.3.37}$$

where $k_2>m$ satisfies

$$3+k_2=(3+m)\frac{\frac{3}{r_2}-1}{1-(m+3)\left(\frac{3}{r_2}-2\right)}=(3+m)\frac{1+\varepsilon/2}{1-(m+3)\,\varepsilon/2}. \tag{5.3.38}$$

Step 2. Estimate for I at infinity

In this step we estimate the norm of I on the exterior of $B = B(0, 3R)$. Observe that when $|x| \geq 3R$ and when $|x - \xi(t-s) - y| \leq R$ we have $|y| \geq |x| - |\xi(t-s)| - R \geq 3R - R_0 - R \geq R_0 - 1 > 1$ (remember that $R_0 > 2$). We use again the parameters $0 < \varepsilon < 2/(m+3)$ and $r_2 = 3/(2 + \varepsilon/2)$. By similar computations we find

$$
\begin{aligned}
&\|I\|_{L^{m+3}(B^c)} \\
&\leq C \Big\| \int_0^t s^{1-\frac{3}{r_2}+\varepsilon} \left(\int_{|x-\xi-y|\leq R} \frac{dy}{|y|^{\varepsilon r_2} |x-\xi-y|^{2r_2}} \right)^{\frac{1}{r_2}} \times \\
&\qquad \times \left(\int (1 + |v|^{\varepsilon r_2'}) f(t-s, X(s), V(s))\, dv \right)^{\frac{1}{r_2'}} \Big\|_{L^{m+3}(B^c)} ds \\
&\leq C \int_0^t s^{1-\frac{3}{r_2}+\varepsilon} \Big\{ \int \left(\int_{|x-\xi-y|\leq R} \frac{dy}{|x-\xi-y|^{2r_2}} \right)^{\frac{m+3}{r_2}} \times \\
&\qquad \times \left(\int dv\ (1 + |v|^{\varepsilon r_2'}) f(t-s, X(s), V(s)) \right)^{\frac{(m+3)}{r_2'}} dx \Big\}^{\frac{1}{m+3}} ds \\
&\leq C \int_0^t s^{1-\frac{3}{r_2}+\varepsilon} \Big\{ \int \left(\int (1 + |v|^{\varepsilon r_2'}) f(t-s, X(s), V(s))\, dv \right)^{\frac{(m+3)}{r_2'}} dx \Big\}^{\frac{1}{m+3}} ds.
\end{aligned}
$$

We now introduce k_3 such that

$$
\left(\frac{3 + \varepsilon r_2'}{3 + k_3} \right) \left(\frac{m+3}{r_2'} \right) = 1.
$$

Note that $k_3 > \varepsilon\, r_2'$ as soon as $m + 3 > r_2'$, which is always true if (like in our case) $m \geq 3$ (remember that $\varepsilon < \frac{2}{m+3}$ so that $r' \leq \frac{3}{1-\frac{1}{m+3}}$). It follows from (5.3.27) with $a = \varepsilon\, r_2'$ and $b = k_3$ that

$$
\|I\|_{L^{m+3}(B^c)} \leq C t^{\frac{3}{r_2}-2} H_{k_3}(t)^{\frac{1}{m+3}}.
$$

Since $k_3 < k_2$, we have $H_{k_3} \leq C H_{k_2}$. Finally, making explicit the dependence of the constant:

$$
\|I\|_{L^{m+3}(B^c)} \leq C(r_2, m)\, t^{\frac{3}{r_2}-2} H_{k_2}(t)^{\frac{1}{m+3}}, \tag{5.3.39}
$$

with $k_2 = \frac{(3+m)\,(3/r_2-1)}{1-(m+3)(3/r_2-2)} - 3$, for all $r_2 \in]\frac{3}{2+\frac{1}{m+3}}, 3/2[$.

Step 3: end of the proof of Proposition 12

Gathering the estimates (5.3.37) and (5.3.39) we find

$$
\left\| \int_0^t s \int_{\mathbb{R}^3} (F_{\text{int}} f)(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}} \leq C(r_2, m) t^{\frac{3}{r_2}-2} H_{k_2}(t)^{\frac{1}{m+3}}, \tag{5.3.40}
$$

hence

$$
\begin{aligned}
\left\| \int_0^t s \int_{\mathbb{R}^3} (E_{\text{int}} + F_{\text{int}}) f(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}} &\leq C(r_1, m)\, t^{2-\frac{3}{r_1}} H_{k_1}(t)^{\frac{1}{m+3}} \\
&\quad + C(r_2, m)\, t^{\frac{3}{r_2}-2} H_{k_2}(t)^{\frac{1}{m+3}}.
\end{aligned} \tag{5.3.41}
$$

We recall that $r_1 \in]3/2, 15/4]$ and $r_2 \in]\frac{3}{2+1/(m+3)}, 3/2[$ can be taken as close as necessary to $3/2$, and that $k_1, k_2 > m$ are defined by $k_1 + 3 = (m+3)(3 - 3/r_1)$ and by $k_2 + 3 = (m+3)(\frac{3}{r_2} - 1)/(1 - (m+3)(3/r_2 - 2))$ (see (5.3.38)).
We next choose r_1 and r_2 so that $k_1 = k_2$ in the following way. We consider a (small) parameter $0 < \gamma < 1$. We define r_1 so that

$$2 - \frac{3}{r_1} = \gamma.$$

Note that $3/2 < r_1 < 3 \leq 15/4$ by choice of γ.
We next define r_2 so that

$$\frac{3}{r_2} - 2 = \frac{\gamma}{1 + (m+3)(\gamma+1)},$$

which implies that $k_2 = k_1$. Then the condition (5.3.35) on r_2 is satisfied. Then $k + 3 = (m+3)(1+\gamma)$, and using that $t \leq 1$, (5.3.41) rewrites

$$\begin{aligned}\left\| \int_0^t s \int_{\mathbb{R}^3} (E_{\text{int}} + F_{\text{int}}) f(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}} &\leq C(\gamma, m) \left(t^{\gamma} + t^{\frac{\gamma}{1+(m+3)(\gamma+1)}} \right) H_k(t)^{\frac{1}{m+3}} \\ &\leq C(\gamma, m)\, t^{\frac{\gamma}{1+(m+3)(\gamma+1)}}\, H_k(t)^{\frac{1}{m+3}} \\ &\leq C(\gamma, m)\, t^{\delta}\, H_k(t)^{\frac{1}{m+3}}.\end{aligned}$$

The conclusion follows. □

Proposition 13 (Intermediate small time estimates)**.** *Let (f, ξ) be a classical, compactly supported solution of (5.3.1)-(5.3.3) on $[0, T]$. For $t \leq \inf(1, T)$ and $3 < m < m_0$, the following estimate holds:*

$$\begin{aligned}\|E(t)\|_{m+3} &\leq C + C \left\| \int_0^t s \int_{\mathbb{R}^3} (E_{\text{int}} + F_{\text{int}}) f(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}} \\ &\leq C + Ct^{\delta} + Ct^{1+\gamma+\delta} H_m(t)^{\frac{3(k+3)}{(m+3)^2}}.\end{aligned} \tag{5.3.42}$$

Here γ is any number in $]0, 1[$ if $m \geq 6$, and any number in $]0, 1[$ such that $\gamma \leq (m-3)/(6-m)$ if $m < 6$. The parameter $k > m$ is defined by $k + 3 = (m+3)(1+\gamma)$, and $\delta = \frac{\gamma}{1+(m+3)(\gamma+1)}$.

Remark 10. *Here we only need that $m_0 > 3$.*

Remark 11. *We stress that the constants depend on k, or equivalently, on γ (in fact some of them blow up when $k \to m$).*

Proof. Thanks to Propositions 11 and 12, we obtain

$$\|E(t)\|_{L^{m+3}} \leq C + C \left\| \int_0^t s \int_{\mathbb{R}^3} (E_{\text{int}} + F_{\text{int}}) f(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}} \leq C + Ct^{\delta}\, H_k(t)^{\frac{1}{m+3}}, \tag{5.3.43}$$

with $k > m$ such that $k + 3 = (m+3)(1+\gamma)$ and $\delta = \frac{\gamma}{1+(m+3)(\gamma+1)}$, and for all $0 < \gamma < 1$.

On the other hand, we infer from Lemma 9 and from Proposition 8 that

$$H_k(t)^{\frac{1}{m+3}} \leq C \left(H_k(0)^{\frac{1}{m+3}} + t^{\frac{k+3}{m+3}} \sup_{s\in[0,t]} \|E(s)\|_{L^{k+3}}^{\frac{k+3}{m+3}} + 1 \right).$$

Therefore, if $k < m_0$, we get

$$H_k(t)^{\frac{1}{m+3}} \leq C \left(1 + t^{\frac{k+3}{m+3}} \sup_{s\in[0,t]} \|E(s)\|_{L^{k+3}}^{\frac{k+3}{m+3}} \right). \tag{5.3.44}$$

Next, by Proposition 10, we have $\|E(s)\|_{L^{3(m+3)/(6-m)}} \leq CH_m(s)^{\frac{3}{m+3}}$ if $m < 6$, and $\|E(s)\|_{L^\infty} \leq CH_m(s)^{\frac{3}{m+3}}$ if $m > 6$. On the other hand, since $m > 3$, one can choose γ sufficiently small in terms of m such that (when $m < 6$) $k + 3 \leq 3(m + 3)/(6 - m)$, namely

$$\gamma \leq \frac{m-3}{6-m}. \tag{5.3.45}$$

By interpolation, this yields

$$\|E(s)\|_{L^{k+3}} \leq CH_m(s)^{\frac{3}{3+m}}. \tag{5.3.46}$$

Therefore, we infer from (5.3.43), (5.3.44) and (5.3.46), that

$$\begin{aligned} \|E(t)\|_{m+3} &\leq C + C \left\| \int_0^t s\, ds \int_{\mathbb{R}^3} (E_{\text{int}} + F_{\text{int}}) f(t-s, X(s), V(s))\, dv \right\|_{L^{m+3}} \\ &\leq C + Ct^\delta + Ct^{1+\gamma+\delta} H_m(t)^{\frac{3(k+3)}{(m+3)^2}}. \end{aligned} \tag{5.3.47}$$

This completes the proof of Proposition 13. □

5.3.6 Bound on the moments

This paragraph is devoted to the proof of the propagation of the moments, formulated in the following

Proposition 14. *Let (f, ξ) be a classical, compactly supported solution of* (5.3.1)-(5.3.3) *on* $[0, T]$. *Then we have for all* $m \in]16/3, \min(m_0, 7)[$

$$H_m(T) \leq C.$$

The constant C depends only on the quantities $\mathcal{H}(0)$, $\mathcal{M}_0$, $\|f_0\|_\infty$, ξ_0, $H_m(0)$, for $m < m_0$, and T.

Proof. Let $t \in [0, T]$. In view of Lemma 9 and Proposition 11 it is enough to control the quantity $\left\| \int_0^t s \int_{\mathbb{R}^3} |E_{\text{int}} + F_{\text{int}}| f(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}}$ in terms of $H_m^{1/(m+3)}$. Unfortunately, the bound obtained in Proposition 13 does not allow to conclude, since it provides an exponent $3(k + 3)/(m + 3)^2$, which is much too large. In order to bypass this difficulty, we shall use, as in [7], two kinds of estimates: for small times we will use the estimate of

Proposition 13; note indeed that the right-hand side is small when t is small. On the other hand for large times we will perform other estimates.
More precisely, let $0 < t_0 < \inf(1,T)$ sufficiently small, to be determined later on. Let $\alpha \in]0,1/4[$.

First case: $t \in [t_0,T]$.

We have

$$\begin{aligned}
&\left\| \int_{t_0}^t s \int (F_{\rm int} f)(t-s,X(s),V(s))\,dv\,ds \right\|_{L^{m+3}} \\
&\le \left\| \int_{t_0}^t s \Big\{ \int |F_{\rm int}|^{3/2-\alpha}(t-s,X(s))\,dv \Big\}^{\frac{2}{3-2\alpha}} \Big\{ \int f^{\frac{3-2\alpha}{1-2\alpha}}(t-s,X(s),V(s))\,dv \Big\}^{\frac{1-2\alpha}{3-2\alpha}} ds \right\|_{L^{m+3}} \\
&\le C \int_{t_0}^t s \Big\{ \int \chi_R(y) \frac{1}{|y|^{3-2\alpha}} \frac{dy}{s^3} \Big\}^{\frac{2}{3-2\alpha}} ds \sup_{\tau,\tau'\in[0,t]} \left\| \int f^{\frac{3-2\alpha}{1-2\alpha}}(\tau,X(\tau'),V(\tau'))\,dv \right\|_{L^{(m+3)\frac{1-2\alpha}{3-2\alpha}}}^{\frac{1-2\alpha}{3-2\alpha}} \\
&\le C \|f_0\|_{L^\infty}^{\frac{2}{3-2\alpha}} \int_{t_0}^t s^{1-\frac{6}{3-2\alpha}}\,ds \sup_{\tau,\tau'\in[0,t]} \left\| \int f(\tau,X(\tau'),V(\tau'))\,dv \right\|_{L^{(m+3)\frac{1-2\alpha}{3-2\alpha}}}^{\frac{1-2\alpha}{3-2\alpha}}.
\end{aligned}$$

We now use (5.3.27) with $a=0$ and b such that $\frac{b+3}{3} = (m+3)\frac{1-2\alpha}{3-2\alpha}$. Note that $b>0$ since $\alpha \in]0,1/4[$ and $m>2$.
We obtain

$$\left\| \int_{t_0}^t s \int (F_{\rm int} f)(t-s,X(s),V(s))\,dv\,ds \right\|_{L^{m+3}} \le C\, t_0^{-\frac{4\alpha}{3-2\alpha}}\, H_{\frac{m-\alpha(2m+4)}{1-\frac{2}{3}\alpha}}(t)^{\frac{1}{m+3}}.$$

We now apply the interpolation inequality

$$H_\beta(t) \le H_2(t)^{\frac{m-\beta}{m-2}} H_m(t)^{\frac{\beta-2}{m-2}}, \quad \beta \in [2,m], \tag{5.3.48}$$

with the choice $\beta = \frac{m-\alpha(2m+4)}{1-\frac{2}{3}\alpha}$. Note that $\beta \in [2,m]$ since $m \ge 16/3$ and $\alpha < 1/4$. Since $\sup_{t\in[0,T]} H_2(t) \le C$ thanks to the conservation of energy, this yields

$$\left\| \int_{t_0}^t s \int (F_{\rm int} f)(t-s,X(s),V(s))\,dv\,ds \right\|_{L^{m+3}} \le C\, t_0^{-\frac{4\alpha}{3-2\alpha}}\, H_m(t)^{\frac{1}{m+3}-\frac{4\alpha}{(3-2\alpha)(m-2)}}. \tag{5.3.49}$$

We emphasize that the constant above depends on α and R.

We obtain an analogous estimate for the internal part of the electric field. Since ρ belongs to $L^\infty([0,T],L^{5/3}(\mathbb{R}^3))$ the internal part $E_{\rm int}$ is bounded in $L^\infty([0,T],L^{3/2-\alpha}(\mathbb{R}^3))$ for all $0 \le \alpha < 1/4$. So by exactly the same computations as before we get

$$\begin{aligned}
&\left\| \int_{t_0}^t s \int (E_{\rm int} f)(t-s,X(s),V(s))\,dv\,ds \right\|_{L^{m+3}} \\
&\le \int_{t_0}^t s^{1-\frac{6}{(3-2\alpha)}} \sup_{\tau,\tau'\in[0,t]} \left(\|E_{\rm int}(\tau)\|_{3/2-\alpha} \left\| \int f(\tau,X(\tau'),V(\tau'))\,dv \right\|_{L^{(m+3)\left(\frac{1-2\alpha}{3-2\alpha}\right)}}^{\left(\frac{1-2\alpha}{3-2\alpha}\right)} \|f_0\|_{L^\infty}^{2/(3-2\alpha)} \right) \\
&\le C\, t_0^{-\frac{4\alpha}{3-2\alpha}}\, H_m(t)^{\frac{1}{m+3}-\frac{4\alpha}{(3-2\alpha)(m-2)}}.
\end{aligned} \tag{5.3.50}$$

Combining (5.3.49) and (5.3.50), we are led to (for all $\alpha \in]0, 1/4[$ and $m \in]16/3, \min(m_0, 7)[$),

$$\left\| \int_{t_0}^{t} s \int (E_{\text{int}} + F_{\text{int}}) f(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}} \leq C\, t_0^{-\frac{4\alpha}{3-2\alpha}}\, H_m(t)^{\frac{1}{m+3} - \frac{4\alpha}{(3-2\alpha)(m-2)}}. \tag{5.3.51}$$

Second case: $t \in [0, t_0]$.

By Proposition 13, we have

$$\left\| \int_{0}^{t} s \int (E_{\text{int}} + F_{\text{int}}) f(t-s, X(s), V(s))\, dv\, ds \right\|_{L^{m+3}} \leq C\,(1 + t^{\delta} + t^{1+\gamma+\delta} H_m(t)^{\frac{3(k+3)}{(m+3)^2}}), \tag{5.3.52}$$

where $k+3 = (m+3)(1+\gamma)$ and where $\delta = \frac{\gamma}{1+(m+3)(\gamma+1)}$, with any $\gamma \in]0,1[$ such that $\gamma \leq \frac{m-3}{6-m}$ if $m > 6$.

Remembering that $t_0 \leq 1$, we deduce from (5.3.51), (5.3.52) and Proposition 11 that for any $t \in [0, T]$,

$$\|E(t)\|_{L^{m+3}} \leq C + C\, t_0^{\delta} + C t_0^{1+\gamma+\delta} H_m(t)^{\frac{3(k+3)}{(m+3)^2}} + C\, t_0^{-\frac{4\alpha}{3-2\alpha}}\, H_m(t)^{\frac{1}{m+3} - \frac{4\alpha}{(3-2\alpha)(m-2)}}.$$

Invoking again Lemma 9 and using that $H_m(t) \geq 1$ and $t_0 \leq 1$ we therefore obtain

$$\begin{aligned} &\frac{d}{dt}\tilde{H}_m(t) \\ &\leq C\left(\|E(t)\|_{L^{m+3}} + |E(t,\xi(t))|\right) H_m(t)^{\frac{m+2}{m+3}} \\ &\leq C\Bigg(1 + t_0^{\delta} H_m^{-\frac{1}{m+3}} + t_0^{1+\gamma+\delta} H_m(t)^{\frac{3(k+3)}{(m+3)^2} - \frac{1}{m+3}} + t_0^{-\frac{4\alpha}{3-2\alpha}}\, H_m(t)^{-\frac{4\alpha}{(3-2\alpha)(m-2)}} + \\ &\qquad + |E(t,\xi(t))| H_m(t)^{-\frac{1}{m+3}}\Bigg) H_m(t). \end{aligned} \tag{5.3.53}$$

We then specify our choice for t_0: we set (for example)

$$t_0^{1+\gamma+\delta} = H_m(t)^{-\frac{3(k+3)-(m+3)}{(m+3)^2}}$$

hence

$$t_0 = H_m(t)^{-\frac{3(k+3)-(m+3)}{(1+\gamma+\delta)(m+3)^2}}. \tag{5.3.54}$$

It follows that

$$t_0^{-\frac{4\alpha}{3-2\alpha}}\, H_m(t)^{-\frac{4\alpha}{(3-2\alpha)(m-2)}} = H_m(t)^{\frac{4\alpha}{3-2\alpha}\left(\frac{3(k+3)-(m+3)}{(1+\delta+\gamma)(m+3)^2} - \frac{1}{m-2}\right)} = H_m(t)^{e(m)},$$

where we denote by $e(m)$ the term appearing in the exponent above,

$$e(m) = \frac{4\alpha}{3-2\alpha}\left(\frac{3(k+3)-(m+3)}{(1+\delta+\gamma)(m+3)^3} - \frac{1}{m-2}\right) = \frac{4\alpha}{3-2\alpha}\left(\frac{2+3\gamma}{(1+\delta+\gamma)(m+3)} - \frac{1}{m-2}\right).$$

We recall that $\gamma > 0$ is a parameter that can be chosen as small as wanted such that the condition (5.3.45) is satisfied, and that $\delta = \gamma/(1 + (\gamma + 1)(m + 3))$.
We now use the assumption $m < 7$. Then

$$\frac{2}{m+3} - \frac{1}{m-2} < 0,$$

and we can choose $\gamma > 0$ sufficiently small, so that $e(m) \leq 0$, and we obtain

$$H_m(t)^{e(m)} \leq 1.$$

Coming back to (5.3.53), we infer that for all $t \in [0, T]$,

$$\frac{d}{dt}\tilde{H}_m \leq C\Big(1 + |E(t,\xi)|\, H_m^{-\frac{1}{m+3}}\Big)\, H_m(t).$$

By a Gronwall argument using Proposition 8, we conclude the proof of Proposition 14. $\square$

5.4 Appendix (estimates for the almost-free flow)

We can write the implicit solution of (5.3.25) as follows:

$$\begin{aligned} f(t,x,v) = &\int_0^t -[(E_{\text{int}} + F_{\text{int}})\,\nabla_v f](t-s, X(s,x,v), V(s,x,v))\, ds \\ &+ f_0(X(t,x,v), V(t,x,v)). \end{aligned} \tag{5.4.1}$$

Computing (for any smooth function g)

$$\begin{aligned} \nabla_x\{x \mapsto g(t-s, X(s,x,v), V(s,x,v))\} = &\nabla_x X(s,x,v)\, \nabla_x g(t-s, X(s,x,v), V(s,x,v)) \\ &+ \nabla_x V(s,x,v)\, \nabla_v g(t-s, X(s,x,v), V(s,x,v))\ , \end{aligned}$$

$$\begin{aligned} \nabla_v\{v \mapsto g(t-s, X(s,x,v), V(s,x,v))\} = &\nabla_v X(s,x,v)\, \nabla_x g(t-s, X(s,x,v), V(s,x,v)) \\ &+ \nabla_v V(s,x,v)\, \nabla_v g(t-s, X(s,x,v), V(s,x,v))\ , \end{aligned}$$

so that (provided that $\nabla_x X$ is invertible and that $\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)$ is also invertible)

$$\begin{aligned} &\nabla_v g(t-s, X(s,x,v), V(s,x,v)) = \\ &\quad = [\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)]^{-1} \\ &\qquad (\nabla_v\{v \mapsto g(t-s, X(s,x,v), V(s,x,v))\} \\ &\qquad - (\nabla_v X)\,(\nabla_x X)^{-1}\nabla_x\{x \mapsto g(t-s, X(s,x,v), V(s,x,v))\}) \\ &\quad = \mathrm{div}_v\{v \mapsto [\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)]^{-1} g(t-s, X(s,x,v), V(s,x,v))\} \\ &\qquad - \mathrm{div}_x\{x \mapsto [\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)]^{-1}(\nabla_v X)\,(\nabla_x X)^{-1} g(t-s, X(s,x,v), V(s,x,v))\} \\ &\qquad - g(t-s, X(s,x,v), V(s,x,v)) \Big\{ \mathrm{div}_v([\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)]^{-1}) \\ &\qquad - \mathrm{div}_x([\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)]^{-1}(\nabla_v X)\,(\nabla_x X)^{-1}) \Big\}. \end{aligned}$$

Then

$$\rho(t,x) = \nabla_x \cdot \int\int [\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)]^{-1}(\nabla_v X)\,(\nabla_x X)^{-1} \tag{5.4.2}$$

$$[(E_{int} + F_{int})\,f](t-s, X(-s,x,v), V(-s,x,v))\,dvds$$

$$+\int\int \Big\{ \nabla_v([\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)]^{-1})$$

$$-\nabla_x([\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)]^{-1}(\nabla_v X)\,(\nabla_x X)^{-1})\Big\}$$

$$[(E_{int} + F_{int})\,f](t-s, X(-s,x,v), V(-s,x,v))\,dvds.$$

We now observe that

$$\frac{d}{ds}\{\nabla_v X(s)\} = \nabla_v V(s), \qquad \nabla_v X(0) = 0,$$

$$\frac{d}{ds}\{\nabla_v V(s)\} = \left(\frac{\partial E_{ext}}{\partial x} + \frac{\partial F_{ext}}{\partial x}\right)(s, X(s))\,\nabla_v X(s), \qquad \nabla_v V(0) = Id,$$

so that, recalling (5.3.23), (5.3.24) and using Gronwall's lemma:

$$||\nabla_v X||_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}, ||\nabla_v X||_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \le C_T. \tag{5.4.3}$$

The same argument used for x-derivatives ensures that

$$||\nabla_x X||_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}, ||\nabla_x V||_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \le C_T. \tag{5.4.4}$$

Then, using (5.4.3), (5.4.4) and (5.3.23), (5.3.24), for any $t \in [0,T]$,

$$|\nabla_x V(t)| = |\int_0^t (\frac{\partial E_{ext}}{\partial x} + \frac{\partial F_{ext}}{\partial x})(s, X(s))\,\nabla_x X(s)\,ds| \le \frac{C_T}{R^2}\,t, \tag{5.4.5}$$

$$|\nabla_v V(t) - Id| = |\int_0^t (\frac{\partial E_{ext}}{\partial x} + \frac{\partial F_{ext}}{\partial x})(s, X(s))\,\nabla_v X(s)\,ds| \le \frac{C_T}{R^2}\,t, \tag{5.4.6}$$

$$|\nabla_x X(t) - Id| = |\int_0^t \int_0^s (\frac{\partial E_{ext}}{\partial x} + \frac{\partial F_{ext}}{\partial x})(\sigma, X(\sigma))\,\nabla_x X(\sigma)\,d\sigma\,ds| \le \frac{C_T}{R^2}\,t^2, \tag{5.4.7}$$

$$|\nabla_v X(t) - t\,Id| = |\int_0^t \int_0^s (\frac{\partial E_{ext}}{\partial x} + \frac{\partial F_{ext}}{\partial x})(\sigma, X(\sigma))\,\nabla_v X(\sigma)\,d\sigma\,ds| \le \frac{C_T}{R^2}\,t^2. \tag{5.4.8}$$

This means that

$$\nabla_v X(s) = -s\,(Id + P(s)), \quad \|P(s)\|_{L^\infty} \le Cs/R^2,$$

and it follows that

$$|\det(\nabla_v X(s))|^{-1} = s^{-3}|\det(I_d + P(s))|^{-1},$$

with $|\det(I_d + P(s))| \geq 1/2$ if R is sufficiently large, so that

$$|\det(\nabla_v X(s))|^{-1} \leq C s^{-3}. \tag{5.4.9}$$

For a given $T > 0$ and all $R > 0$ large enough, we deduce from (5.4.5) – (5.4.8) that $\nabla_x X$ is indeed invertible, and so is $\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)$. As a consequence, eq. (5.4.2) holds, and (for $s \in [0,T]$)

$$||[\nabla_v V(s,\cdot,\cdot) - (\nabla_v X)(s,\cdot,\cdot)\,(\nabla_x X)^{-1}(s,\cdot,\cdot)(\nabla_x V)(s,\cdot,\cdot)]^{-1}(\nabla_v X)(s,\cdot,\cdot)(\nabla_x X)^{-1}(s,\cdot,\cdot) \tag{5.4.10}$$

$$-s\,Id||_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \leq \frac{C_T}{R^2}\,s^2,$$

$$||(x,v) \mapsto X(s) - (x + v\,s)||_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \leq \frac{C_T}{R^2}\,s^2, \tag{5.4.11}$$

$$||(x,v) \mapsto V(s) - v||_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \leq \frac{C_T}{R^2}\,s^2. \tag{5.4.12}$$

Finally, writing the differential system satisfied by the second derivatives w.r.t. x, v of X, V and using Gronwall's lemma, it is possible to show that for any such second derivative D^2,

$$||D^2 X||_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}, ||D^2 V||_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \leq C_T,$$

and, as a consequence, for any given $T > 0$ and $R > 0$ large enough

$$||\nabla_v([\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)]^{-1}) \tag{5.4.13}$$

$$-\nabla_x([\nabla_v V - (\nabla_v X)\,(\nabla_x X)^{-1}(\nabla_x V)]^{-1}(\nabla_v X)\,(\nabla_x X)^{-1})||_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \leq C_T.$$

Bibliography

[1] L. Ambrosio and G. Crippa, *Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields*, Transport Equations and Multi-D Hyperbolic Conservation Laws, Lecture Notes of the Unione Matematica Italiana, 2008.

[2] A. A. Arsenev, *Global existence of a weak solution of Vlasov's system of equations*, U. S. S. R. Comput. Math. Math. Phys. **15** (1975), 131–143.

[3] S. Caprino and C. Marchioro, *On the plasma-charge model*, Kinet. Relat. Models **3** (2) (2010), 241–254.

[4] R. J. Di Perna and P. L. Lions, *Ordinary differential equations, transport equations and Sobolev spaces*, Inv. Math. **98** (1989), 511–547.

[5] I. Gasser, P. E. Jabin and B. Perthame, *Regularity and propagation of moments in some nonlinear Vlasov systems*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), 1259–1273.

[6] M. Hauray, *On Liouville transport equation with a force field in BV_{loc}*, Comm. Partial Differential Equations **29** (2004), no. 1–2, 207–217.

[7] P. L. Lions and B. Perthame, *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system*, Invent. Math. **105** (1991), 415–430.

[8] G. Loeper, *Uniqueness of the solution to the Vlasov-Poisson system with bounded density*, J. Math. Pures Appl. (9) **86** (2006), no. 1, 68–79.

[9] S. Okabe, T. Ukai, *On classical solutions in the large in time of the two-dimensional Vlasov equation*, Osaka J. Math. **15** (1978), 245–261.

[10] C. Pallard, *Moment propagation for weak solutions to the Vlasov-Poisson system*, preprint, 2011.

[11] K. Pfaffelmoser, *Global existence of the Vlasov-Poisson system in three dimensions for general initial data*, J. Differ. Equ. **95** (1992), 281–303.

[12] C. Marchioro, E. Miot and M. Pulvirenti, *The Cauchy problem for the* $3-D$ *Vlasov-Poisson system with point charges*, Arch. Ration. Mech. Anal. **201** (2011), 1-26.

[13] J. Schaeffer, *Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions*, Commun. Partial Differ. Equ. **16** (8–9) (1991), 1313–1335.

[14] S. Wollman, *Global in time solution to the three-dimensional Vlasov-Poisson system*, J. Math. Anal. Appl. **176** (1) (1996), 76–91.

Bibliography

[Bal] R. Balescu, "Equilibrium and Nonequilibrium Statistical Mechanics". John Wiley & Sons, New-York, 1975.

[BPS] A.V. Bobylev, M. Pulvirenti and C. Saffirio, *From particle system to the Landau equation: a consistency result.* Comm. Math. Phys., DOI 10.1007/s00220-012-1633-6 (2012)

[B] L. Boltzmann, *Lectures on gas theory.* English edition annotated by S. Brush, University of California Press, Berkeley, 1964 (reprint).

[CM] E. Caglioti and C. Maffei, *Time asymptotics for solutions of Vlasov-Poisson equation in a circle.* J. Statist. Phys. **92** (1998), 301-323

[CIP] C. Cercignani, R. Illner and M. Pulvirenti, "The Mathematical Theory of Dilute Gases". *Applied Mathematical Sciences* **106**, Springer–Verlag, New York, 1994.

[DMS] L. Desvillettes, E. Miot and C. Saffirio, *An existence result for the repulsive 3d plasma-charge model.* Draft (2012).

[D-79] R. L. Dobrushin, *Vlasov equations.* Sov. J. Funct. Anal. 13 (1979).

[HJ] M. Hauray and P.E. Jabin, *N-particles approximation of the Vlasov equations with singular potential.* Arch. Ration. Mech. Anal. **183**, 3 (2007), 489–524.

[K] M. Kac, *Foundations of kinetic theory.* "Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability", University of California Press, Berkeley and Los Angeles (1956).

[K-75] F. King, *BBGKY Hierarchy for Positive Potentials.* Ph.D. Thesis, Department of Mathematics, Univ. California, Berkeley, 1975.

[L-36] L.D. Landau, *Kinetic equation in the case of Coulomb interaction.* Phys. Zs. Sow. Union, **10** (1936), s. 154.

[L-75] O. E. Lanford, *Time evolution of large classical systems.* "Dynamical systems, theory and applications", *Lecture Notes in Physics*, ed. J. Moser, **38**, 1–111, Springer–Verlag, Berlin, 1975.

[LL] E. M. Lifshitz and L. P. Pitaevskii, "Course of theoretical physics Landau-Lifshits". Pergamon Press, Oxford-Elmsford, N.Y. 1981.

[LP] P. L. Lions and B. Perthame, *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system.* Invent. Math. **105** (1991), 415–430.

[MMP] C. Marchioro, E. Miot and M. Pulvirenti, *The Cauchy problem for the* $3-D$ *Vlasov-Poisson system with point charges.* Arch. Ration. Mech. Anal. **201** (2011), 1-26.

[MPS] E. Miot, M. Pulvirenti and C. Saffirio, *On the Kac model for the Landau equation.* Kinet. Relat. Models 4, No. 1, (2011), 333–344.

[MV] C. Mouhot and C. Villani, *On Landau damping.* Acta Math. **207** (2011), 29–201

[P] M. Pulvirenti, *The weak-coupling limit of large classical and quantum systems.* (English summary) in "International Congress of Mathematicians", Eur. Math. Soc., Zürich (2006).

[PSS] M. Pulvirenti, C. Saffirio and S. Simonella, *On the validity of the Boltzmann equation for short range potentials.* Preprint (2012).

[TH] D. Ter Haar, "Collected papers of L.D. Landau". Physical review, N.Y.1965.

[V-98] C. Villani, *On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations.* Arch. Rational Mech. Anal., **143** (1998), 273–307.

[V-02] C. Villani, *A review of mathematical topics in collisional kinetic theory.* "Handbook of mathematical fluid dynamics", Vol. I, 71–305, North-Holland, Amsterdam, 2002.